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A THEOREM ON $M_{k,m}$ TRANSFORM

By

S. K. KULSHRESHTHA

**Department of Mathematics, Rajasthan University, Jaipur, India*

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ABSTRACT

The object of this paper is to evaluate an integral involving confluent hypergeometric functions. The result obtained comes in double generalised hypergeometric series, a special case of the series given by Kampe de Feriet. With the help of this integral one theorem is also obtained. One infinite integral involving double hypergeometric series $F^{(2)}$ is also obtained with the help of this theorem.

1. INTRODUCTION

The Laplace Transform

$$\psi(p) = p \int_0^\infty e^{-pt}\, h(t)\, dt \qquad \ldots\ldots (1\cdot1)$$

has been generalized in the form[5]

$$\phi(p) = p \int_0^\infty (2pt)^{-\frac{1}{2}}\, M_{k,m}\, (2pt)\, h(t)\, dt. \qquad \ldots\ldots (1\cdot2)$$

Equation (1·2) reduces to (1·1) when $k = -m = \frac{1}{4}$ due to the identity

$$M_{\frac{1}{4}, -\frac{1}{4}}(x) = x^{\frac{1}{4}}\, e^{-x/2} \qquad \ldots\ldots (1\cdot3)$$

We shall denote the integral equation (1·2) symbolically as

$$\phi(p) \underset{k,\, m}{\overset{M}{=}} h(t)$$

and the equation (1·1) as usual shall be denoted as

$$\psi(p) \doteqdot h(t).$$

**Present address*: Department of Mathematics, S. A. T. I. Vidisha, (M. P.)

The object of this paper is to evaluate an integral involving confluent hypergeometric functions and with the help of this integral to obtain a theorem connecting $M_{k,m}$ Transform and Laplace Transform. Some infinite integrals are also evaluated with the help of the theorem.

2. Theorem. If

$$\phi(p) \underset{k,\mu}{\overset{M}{=}} t^{\sigma} M_{\lambda,\nu}(\alpha t)\, \psi(t)$$

and

$$\psi(p) \doteq h(t)$$

then

$$\phi(p) = p\,(2p)^{\mu+\frac{1}{2}} \alpha^{\nu+\frac{1}{2}} 2^{\sigma+\mu+\nu+11/4}\, \Gamma(11/4+\mu+\nu+\sigma) \int_0^{\infty} (2t+2p-\alpha)^{-11/4-\mu-\nu-\sigma} h(t) \times$$

$$F^{(2)}\left[11/4+\mu+\nu+\sigma\,;\tfrac{1}{2}-k+\mu,\tfrac{1}{2}+\lambda+\nu;\,2\mu+1,2\nu+1\,;\frac{4p}{2t+2p-\alpha},\frac{-2\alpha}{2t+2p-\alpha}\right] dt \quad \ldots\ldots \quad (2\cdot1)$$

provided $R(\alpha) > R(p) > 0$, $R(11/4+\mu+\nu+\sigma) > 0$.
and the integral involved in (2·1) is absolutely convergent.

Before proving the above theorem we shall evaluate the following integral :

$$\int_0^{\infty} t^{\rho-1} e^{-\frac{1}{2}(\alpha+\beta-2\gamma)} M_{\lambda,\nu}(\alpha t)\, W_{\eta,\xi}(\beta t)\, M_{k,\mu}(2\gamma t)\, dt$$

$$= \frac{\alpha^{\nu+\frac{1}{2}} (2\gamma)^{\mu+\frac{1}{2}}\, \Gamma(3/2+\nu+\rho+\mu\pm\xi)}{\beta^{\nu+\mu+\rho+1}\, \Gamma(2-\eta+\nu+\mu+\rho)} F^{(6)}\left[\begin{matrix}3/2+\nu+\rho+\mu\pm\xi : \frac{1}{2}-k+\mu,\ \frac{1}{2}+\lambda+\nu \\ 2-\eta+\nu+\mu+\rho : 2\mu+1,\ 2\nu+1\end{matrix}\,;\,\frac{2\gamma}{\beta},\,\frac{-\alpha}{\beta}\right] \ldots\ldots \quad (2\cdot2)$$

where $R(\beta) > R(2\gamma) > 0$, $R(\alpha) > 0$, $R(\rho+\mu+\nu+\frac{1}{2}) > 0$, and the double generalized hypergeometric series

$$F^{(6)}\left[\begin{matrix}a,\ b : d,\ e \\ c : f,\ g\end{matrix}\,;x,y\right] = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(a)_{r+s}\,(b)_{r+s}\,(d)_r\,(e)_s}{(c)_{r+s}\,(f)_r\,(g)_s\,\lfloor r\,\lfloor s} x^r y^s \quad \ldots\ldots \quad (2\cdot3)$$

is a special case of the Kampe de Feriet's hypergeometric function of two variables of higher order and symbolically denoted in Kampe de Feriet's notation as

$$\left[\begin{array}{c|c|c} 2 & c,\ d & \\ 1 & a\,,\ b & x \\ 1 & e & \\ 1 & a',\ b' & y \end{array}\right]$$

Now in L. H. S. of integral (2·2) changing the $M_{k,\mu}(2\gamma t)$ into the ${}_1F_1$ function, we get

$$(2\gamma)^{\mu+\frac{1}{2}} \int_0^{\infty} t^{\rho+\mu-\frac{1}{2}} e^{-\frac{1}{2}(\alpha+\beta)\,t} M_{\lambda,\nu}(\alpha t)\ W_{\eta,\xi}(\beta t)\ {}_1F_1\left[\begin{matrix}\frac{1}{2}-k+\mu \\ 2\mu+1\end{matrix}\,;\,2\gamma t\right] dt$$

$$= (2\gamma)^{\mu+\frac{1}{2}} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}-k+\mu)_r}{\lfloor r\,(2\mu+1)_r} (2\gamma)^r \int_0^{\infty} t^{\rho+\mu+r-\frac{1}{2}} e^{-\frac{1}{2}(\alpha+\beta)t} M_{\lambda,\nu}(\alpha t)\, W_{\eta,\xi}(\beta t)\, dt \quad \ldots\ldots \quad (A)$$

$$= I$$

Solving this integral with the help of result [3, p. 410] we get

$$I = \frac{\alpha^{\nu+\frac{1}{2}} (2\gamma)^{\mu+\frac{1}{2}}}{\beta^{\rho+\mu+\nu+1}} \sum_{r=0}^{\infty} \frac{\Gamma(\frac{3}{2}+\nu+\rho+\mu\pm\xi+r)\,(\frac{1}{2}-k+\mu)_r}{\lfloor r \;\; \Gamma(2-\eta+\nu+\rho+\mu+r)\,(2\mu+1)_r}$$

$$(2\gamma/\beta)^r \; {}_3F_2 \left[\begin{matrix} \frac{1}{2}+\lambda+\nu,\ \frac{3}{2}+\nu+\rho+\mu+\xi+r \\ 2\nu+1,\ 2-\eta+\nu+\rho+\mu+r \end{matrix} ; -\frac{\alpha}{\beta} \right]$$

Again expanding ${}_3F_2$ and on interpreting the result with the help of (2·3) we get the result (2·2).

In (A) the term by term integration is justified since the integral is absolutely convergent for the conditions stated with the result (2·2).

Particular cases:

In the integral (2·2) taking $\eta = \xi + \frac{1}{2}$ and $\sigma = \rho + \xi + \frac{1}{2}$, we get

$$\int_0^{\infty} t^{\sigma-1} e^{-\frac{1}{2}(\alpha+2\beta-2\gamma)t} M_{\lambda,\nu}(\alpha t) M_{k,\mu}(2\gamma t)\, dt$$

$$= \frac{\alpha^{\nu+\frac{1}{2}} (2\gamma)^{\mu+\frac{1}{2}}}{\beta^{\nu+\mu+\sigma+1}} \Gamma(\sigma+\nu+\mu+1)\; F^{(2)}\left[\sigma+\mu+\nu+1 ; \frac{1}{2}-k+\mu, \frac{1}{2}+\lambda+\nu, 2\mu+1, 2\nu+1 ; \frac{2\gamma}{\beta}, \frac{-\alpha}{\beta} \right] \quad \ldots\ldots \quad (2·4)$$

where $R(\alpha) > R(\beta) > 0$, $R(\sigma+\mu+\nu+1) > 0$ and $R(\beta) > R(2\gamma)$

and $F^{(2)}$ is double generalized hypergeometric function [1, p. 224] defined as

$$F^{(2)}[\alpha ; \beta, \beta', \gamma, \gamma' ; x, y] = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\alpha)_{r+s}\,(\beta)_r\,(\beta')_s}{\lfloor r \; \lfloor s \; (\gamma)_r\,(\gamma')_s}\, x^r y^s$$

From the relation (2·4) we get the following operational images

$$t^{\sigma-1} e^{-(\alpha-2\gamma)t/2} M_{\lambda,\nu}(\alpha t) \;\; M_{k,\mu}(2\gamma t)$$

$$\doteqdot \frac{\alpha^{\nu+\frac{1}{2}} (2\gamma)^{\mu+\frac{1}{2}} \Gamma(1+\mu+\nu+\sigma)}{p^{\mu+\nu+\sigma}} \; F^{(2)}\left[\nu+\mu+\sigma+1 ; \frac{1}{2}-k+\mu, \frac{1}{2}+\lambda+\nu, 2\mu+1, 2\nu+1 ; \frac{2\gamma}{p}, \frac{-\alpha}{p} \right] \quad \ldots\ldots \quad (2·5)$$

for $R(\alpha) > R(p) > 0$, $R(\sigma+\mu+\nu+1) > 0$ and $R(p) > R(2\gamma) > 0$.

Again using the operational property that if

$$g(p) \doteqdot h(t)$$

then

$$p(p+a)^{-1} g(p+a) \doteqdot e^{-at} h(t)$$

we can write the relation (2·5) as

$$t^{\sigma-1}\, M_{\lambda,\nu}\,(\alpha t)\, M_{k,\mu}\,(2\gamma t) \doteqdot \frac{p\,\alpha^{\nu+\frac{1}{2}}\,(2\gamma)^{\mu+\frac{1}{2}}\,\Gamma(1+\mu+\nu+\sigma)}{(2p+2\gamma-\alpha)^{\sigma+\mu+\nu+1}} \times 2^{\sigma+\mu+\nu+1}$$

$$F^{(2)}\Big[1+\nu+\sigma+\mu\,;\tfrac{1}{2}-k+\mu,\ \tfrac{1}{2}+\lambda+\nu,\ 2\mu+1,\ 2\nu+1\,;\ \frac{4\gamma}{2p+2\gamma-\alpha},\ \frac{-2\alpha}{2p+2\gamma-\alpha}\Big] \quad \ldots\ldots \quad (2\cdot6)$$

for $R(\alpha) > R(p) > R(\gamma+\frac{1}{2}\alpha) > 0$ and $R(\sigma+\mu+\nu+1) > 0$.

Now with the help of result (2·6) we shall prove the theorem. We have by hypothesis

$$\psi(p(\doteqdot h(t)$$

Now using the relations (2·7) and (2·6) in Parseval Goldstein theorem[4], we get

$$\int_0^{8} t^{\sigma-\frac{3}{4}}\, M_{k,\mu}\,(2\gamma t)\, M_{\lambda,\nu}\,(\alpha t)\,\psi(t)\,dt = \alpha^{\nu+\frac{1}{2}}\,(2\gamma)^{\mu+\frac{1}{2}}\,\Gamma(11/4+\mu+\nu+\sigma)\ \ 2^{\sigma+\mu+\nu+11/4}$$

$$\times \int_0^{\infty} (2t+2\gamma-\alpha)^{-11/4-\mu-\nu-\sigma}\, h(t)\ F^{(2)}\Big[11/4+\mu+\nu+\sigma\,;\tfrac{1}{2}-k+\mu,\ \tfrac{1}{2}+\lambda+\nu,\ 2\mu+1,\ 2\nu+1\,;\frac{2\gamma}{2t+2\gamma-\alpha},\ \frac{-\alpha}{2t+2\gamma-\alpha}\Big]\,dt$$

Multiplying both sides by $\gamma(2\gamma)^{-\frac{3}{4}}$ and finally on replacing γ by p we obtain the theorem.

Corollary—In the theorem if we take $k = -\mu = \frac{1}{4}$ we get

If

$$\phi(p) \doteqdot t^{\sigma}\, M_{\lambda,\nu}\,(\alpha t)\,\psi(t)$$

and

$$\psi(p) \doteqdot h(t)$$

then

$$\phi(p) = p(\alpha)^{\nu+\frac{1}{2}}\ \Gamma(5/2+\nu+\sigma)\ 2^{\sigma+\nu+\frac{5}{2}} \int_0^{\infty} (2t+2p-\alpha)^{-5/2-\nu-\sigma}$$

$${}_2F_1\left[\begin{matrix}5/2+\nu+\sigma,\ \frac{1}{2}+\lambda+\nu\\ 2\nu+1\end{matrix}\,;\ \frac{-2\alpha}{2t+2p-\alpha}\right] h(t)\,dt \qquad \ldots\ldots \quad (2\cdot8)$$

provided $R(\alpha) > R(p) > 0$, $R(\sigma+\nu+5/2) > 0$ and the integral involved in (2·8) is absolutely convergent.

Example—In the theorem if we take

$$h(t) = t^{\rho}$$

$$\doteqdot \Gamma(\rho+1)\ p^{-\rho}$$

$$= \psi(p)$$

where $R(\rho+1) > 0$ and $R(p) > 0$.

Now since by hypothesis

$$\phi(p) \underset{k,\,\mu}{\overset{M}{=}} t^{\sigma} M_{\lambda,\nu}(\alpha t)\, \psi(t)$$

we have on using the relation (2·4)

$$\phi(p) = \frac{p(2p)^{\mu+\frac{3}{4}}\ \Gamma(\rho+1)\ \alpha^{\nu+\frac{1}{2}}\ \Gamma(\frac{7}{4}+\mu+\nu+\sigma-\rho)}{(\gamma-\frac{1}{2}\alpha)^{\frac{7}{4}+\nu+\mu+\sigma-\rho}} \times$$

$$F^{(2)}\left[\tfrac{7}{4}+\mu+\nu+\sigma-\rho,\ ;\ \tfrac{1}{2}-k+\mu,\ \tfrac{1}{2}+\lambda+\nu,\ 2\mu+1,\ 2\nu+1\ ;\ \frac{2p}{\gamma-\frac{1}{2}\alpha},\ \frac{\alpha}{\gamma-\frac{1}{2}\alpha}\right] \quad \ldots . \quad (2\cdot9)$$

Putting the value of $h(t)$ in relation (2·1) and equating this with the value of $\phi(p)$ obtained in relation (2·9) we get

$$\int_0^\infty t^{\rho}\,(t+2p-\alpha)^{-11/4-\mu-\nu-\sigma}\ F^{(2)}\left[11/4+\mu+\nu+\sigma\ ;\ \tfrac{1}{2}-k+\mu,\ \tfrac{1}{2}+\lambda+\nu,\ 2\mu+1,\ 2\mu+1\ ;\ \frac{4p}{t+2p-\alpha},\ \frac{-2\alpha}{t+2p-\alpha}\right] dt$$

$$= \frac{\Gamma(\frac{7}{4}+\mu+\nu+\sigma-\rho)\ \Gamma(\rho+1)}{(2p-\alpha)^{\frac{7}{4}+\mu+\nu+\sigma-\rho}\ \Gamma(11/4+\mu+\nu+\sigma)}\ F^{(2)}\left[\tfrac{7}{4}+\mu+\nu+\sigma-\rho\ ;\ \tfrac{1}{2}-k+\mu,\ \tfrac{1}{2}+\lambda+\nu,\ 2\mu+1,\ 2\nu++1\ ;\ \frac{4p}{2p-\alpha},\ \frac{-2\alpha}{2p-\alpha}\right] \quad \ldots . \quad (2\cdot10)$$

where $R(\rho+1) > 0$, $R(\alpha) > R(p) > 0$ and $R(11/4+\mu+\nu+\sigma) > 0$.

On adjusting the parameters we get

$$\int_0^\infty t^{\rho}\,(t+p)^{-a}\ F^{(2)}\left[a-b-d-c-e\ ;\ \frac{p+\alpha}{t+p} : \frac{p-\alpha}{t+p}\right] dt$$

$$= \frac{\Gamma(a-\rho-1)\ \Gamma(\rho+1)}{p^{a-\rho-1}\ \Gamma(a)}\ F^{(2)}\left[a-\rho-1\ ;\ b,\ d,\ c,\ e\ ;\ \frac{p+\alpha}{p},\ \frac{p-\alpha}{p}\right] \quad \ldots . \quad (2\cdot11)$$

where $R(\rho+1) > 0$, $R(\alpha) > R(p) > 0$ and $R(\alpha) > 0$.

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THEOREMS ON WHITTAKER TRANSFORM AND INFINITE INTEGRALS INVOLVING BESSEL'S FUNCTION

By

S. K. KULSHRESHTHA

**Department of Mathematics, Rajasthan University, Jaipur, India*

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ABSTRACT

In this paper two theorems relating Whittaker Transform defined by Varma R. S. in the form

$$\phi(p) = p \int_0^\infty (2pt)^{-\frac{1}{2}} W_{k,m}(2pt)\, h(t)\, dt$$

with well known Laplace Transform and H- Transform defined as

$$\psi(p) = \int_0^\infty (pt)^{\frac{1}{2}} H_v(pt)\, h(t)\, dt.$$

respectively are obtained. Also with the help of these theorem some infinite integrals involving Hypergeomentric function and Bessel's functions are evaluated.

1. Introduction

The classical Laplace Transform

$$L[h(t)\,;\,]p = p \int_0^\infty e^{-pt} h(t)\, dt \qquad \ldots.(1\cdot1)$$

has been generalised by Verma[5] in the form

$$W[h(t)\,;\,k, m] = p \int_0^\infty (2pt)^{-\frac{1}{2}} W_{k,m}(2pt)\, h(t)\, dt \qquad \ldots.(1\cdot2)$$

where $W_{k,m}(x)$ is Whittaker function.

Relation (1·2) can be reduced to (1·1) due to the identity

$$x^{-\frac{1}{2}} W_{\frac{1}{2},\pm\frac{1}{2}}(x) \equiv e^{-x/2}$$

Also we shall use H–Transform defined by the integral equation

$$H[h(t)\,;\,v] = \int_0^\infty (pt)^{\frac{1}{2}} H_v(pt)\, h(t)\, dt. \qquad \ldots.(1\cdot3)$$

In this paper we shall prove two theorems relating Whittaker Transform with Laplace transform and H·transform respectively. Some infinite integrals are also evaluated with the help of these theorems.

Present address : Department ol Mathematics, S. A. T. I. Vidisha, (M. P.).

2. Theorem 1. If

$$\phi(p) = W[\, t^{\sigma+1}\, h(1/t)\,;\, k, m]$$

and

$$\psi(p, \alpha) = L[\, e^{-\alpha/t}\, h(t)\,;\, p] \qquad \ldots . (21\cdot)$$

then

$$\phi(p) = p(2p)^{\frac{1}{2}} \sum_{m,-m} \frac{\Gamma(-2m) \times (2p)^m}{\Gamma(\frac{1}{2}-k-m)\ \Gamma(13/4+\sigma+m)} \int_0^\infty t^{\sigma+m+5/4}\ \psi(t, p)$$
$${}_1F_2\left[\begin{matrix} \frac{1}{2}-k+m \\ 2m+1,\ 13/4+\sigma+m \end{matrix};\, 2pt\right] dt$$

provided $R(13/4+\sigma \pm m) > o$, $R(p) > o$ and the integrals involved in the relation (2·2) are absolutely convergent and also $h(t)$ is independent of α.

Proof. We have (2. p. 295)

$$p^{-\sigma-7/4}\ e^{\alpha/p}\ W_{k,m}(2\alpha/p) = L\Bigg[(2\alpha)^{\frac{1}{2}}\, t^{\sigma+9/4} \sum_{m,-m} \frac{\Gamma(-2m)\,(2\alpha t)^m}{\Gamma(\frac{1}{2}-k-m)\ \Gamma(13/4+\sigma+m)}$$
$${}_1F_2\left[\begin{matrix} \frac{1}{2}-k+m \\ 2m+1,\ 13/4+\sigma+m \end{matrix};\, 2\alpha t\right];\, p\Bigg]$$

where $R(p) > o$, $R(\alpha) > o$ and $R(13/4+\sigma \pm m) > o \qquad \ldots . (2\cdot3)$

Applying Perseval's Goldstein theorem[4] to the relations (2·1) and (2·3) we get

$$\int_0^\infty t^{-\sigma-11/4} W_{k,m}(2\alpha/t)\, h(t)\, dt =$$
$$(2\alpha)^{\frac{1}{2}} \sum_{m,-m} \frac{\Gamma(-2m)\,(2\alpha)^m}{\Gamma(\frac{1}{2}-k-m)\ \Gamma(13/4+\sigma+m)} \int_0^\infty t^{\sigma+m+5/4}$$
$${}_1F_2\left[\begin{matrix} \frac{1}{2}-k+m \\ 2m+1,\ 13/4+\sigma+m \end{matrix};\, 2\alpha t\right] \psi(t, \alpha)\, dt$$

Replacing t by $1/t$ in the left integral, multiplying both sides by $\alpha\,(2\alpha)^{-\frac{1}{2}}$ and finally on replacing α by p we obtain theorem.

Corollary 1.—In the theorem if we take $\sigma = -k - 11/4$ we get the theorem as

If

$$\phi(p) = W\left[t^{-k-7/4}\ h(1/t)\,;\, k, m\right]$$

and

$$\psi(p, \alpha) = L\left[e^{-\alpha/t}\ h(t)\,;\, p\right]$$

then

$$\phi(p) = \frac{(2p)^{5/4}}{\Gamma(\frac{1}{2}-k\pm m)} \int_0^\infty t^{-k-3/2}\, K_{2m}(2\sqrt{2pt})\, \psi(t, p)\, dt \qquad \ldots . (2\cdot4)$$

provided $R(p) > o$, $R(\alpha) > o$, $R(\frac{1}{2}-k \pm m) > o$, $h(t)$ is independent of α and the integral involved in (2·4) is absolutely convergent.

Corollary 2.—In the corollary 1 if we take $k = \pm m = \frac{1}{4}$ we get

If

$$\phi(p) = L\,[\, t^{-2}\, h\,(1/t)\,;\, p\,]$$

and

$$\psi(p, \alpha) = L\,[\, e^{-\alpha/t}\, h(t)\,;\, p\,]$$

then

$$\phi(p) = p \int_0^\infty t^{-2}\, e^{-2\sqrt{2pt}}\, \psi(t, p)\, dt \qquad \ldots\ldots (2\cdot5)$$

provided $R(p) > o$, $R(\alpha) > o$, $h(t)$ is independent of α and the integral involved in (2·5) is absolutely convergent.

Example. If we take

$$h(t) = t^{\,v-1}\, e^{-q/t}$$

then (2., p. 146)

$$\begin{aligned} e^{-\alpha/t}\, h(t) &= e^{-(\alpha+q)/t}\, t^{\,v-1} \\ &\doteqdot 2\, p^{\,1-v/2}\, (\alpha + q)^{\,v/2}\, K_v\, (2\, \sqrt{p((\alpha+q)}) \\ &= \psi(p, \alpha) \end{aligned}$$

where $(\alpha) > o$ and $R(p) > o$.

Since

$$\phi(p) = W\,[\, t^{\sigma+1}\;\; h(1/t)\,;\, k, m]$$

we have [1., p. 13]

$$\phi\,(p) = \tfrac{1}{2}\cdot \frac{\Gamma(\sigma - v + 13/4 \pm m)}{(2p)^{\,\sigma-\nu+2}\, \Gamma\,(\sigma - \nu - k + 15/4)}\;\; {}_2F_1 \left[\begin{matrix} \sigma - v + 13/4 \pm m \\ \sigma - v - k + 15/4 \end{matrix}\;;\tfrac{1}{2} - \frac{q}{2p} \right] \ldots\ldots (2\cdot6)$$

where $R(\sigma - v + 9/4 \pm m) > o$, $R(p) > R(p_0) > o$ and $|\, p\,| > |\, q\,|$.

Now using the value of $\psi(p, \alpha)$ in relation (2·1) and equating this value of $\phi(p)$ so obtained with the value of $\phi(p)$ obtained in relation (2·6) and on adjusting the parameters wet get

$$\sum_{m,\,-m} \frac{\Gamma(-2m)\,(2p)^m}{\Gamma(\frac{1}{2} - k - m)\;\Gamma(\sigma + m + \frac{v}{2} - 1)} \int_0^\infty t^{\sigma+m-1}\, K_v\,\left(2\sqrt{t\,(p+q)}\right)$$

$${}_1F_2 \left[\begin{matrix} \frac{1}{2} - k + m \\ 2m + 1,\; \sigma + m + \frac{v}{2} - 1 \end{matrix}\;;\, 2pt \right] dt$$

$$= \frac{(p+q)^{-v/2}\;\Gamma(\sigma - v \pm m - 2)}{2\,(2p)^{\,\sigma - v/2 - 2}\;\Gamma(\sigma - \frac{v}{2} - k - 3/2)}\;\; {}_2F_1 \left[\begin{matrix} \sigma - v/2 \pm m - 2 \\ \sigma - v - k - 3/2 \end{matrix}\;;\, \tfrac{1}{2} - \frac{q}{2p} \right] \qquad \ldots\ldots (2\cdot7)$$

where $R(\sigma - v/2 \pm m - 2) > o$, $R(p) > R(p_0) > o$ and $|\, p\,| > |\, q\,|$.

3. *Theorem* 2. If

$$\phi(p) = W[\, t^{\lambda-1}\, \psi(\sqrt{t})\,;\, k, \mu\,]$$

and

$$e^{\alpha p^2}\, \psi(p) = H\,[g\,(t, \alpha)\,;\, v] \qquad \ldots.(3\cdot1)$$

then

$$\phi(p) = \frac{\pi^{-1/2}\, 2^{-3\lambda-3v/2-9/4}\ \Gamma\,(\sigma + \tfrac{v}{2} \pm \mu + 2)}{p^{v/2+\lambda+1/4}\ \Gamma(v + 3/2)\ \Gamma(5/2+\lambda-k+v/2)} \int_0^\infty t^{v+\frac{1}{2}}$$

$${}_3F_3\left[\begin{matrix}1,\, 2 + v/2 + \lambda \pm \mu \\ 3/2,\, v + 3/2,\, 5/2 + \lambda - k + v/2\end{matrix}\,;\, -t^2/8p\right] g(t, p)\, dt \qquad \ldots.(3\cdot2)$$

provided $R(2\lambda + v + 1/2) > 2\,|\,R(\mu)\,| - 7/2$, $R(p) > o$, $\psi(p)$ is indenpendent of α and the integral involved in the relation (3·2) is absolutely convergent.

Proof : We have [3., p. 171]

$$\frac{\pi^{-1/2}\, p^{v+3/2}\, 2^{-3\lambda-3v/2-2}\ \Gamma\,(2 + \lambda + v/2 \pm \mu)}{\lambda^{+v/2+1}\ \Gamma\,(v + 3/2)\ \Gamma(5/2 + \lambda - k - v/2)}$$

$${}_3F_3\left[\begin{matrix}1,\, v/2 + \lambda \pm \mu + 2 \\ 3/2,\, v + 3/2,\, 5/2 + \lambda - k + v/2\end{matrix}\,;\, -p^2/8\alpha\right]$$

$$= H\,[\,t^{2\lambda+\frac{1}{2}}\, e^{-\alpha t^2}\, W_{k,\mu}\,(2\alpha\, t^2)\,;\, v] \qquad \ldots.(3\cdot3)$$

where $R(2\,\lambda + v + \frac{1}{2}) > 2\,|\,R(\mu)\,| - 7/2$, $R(p) > o$, $R(\alpha) > o$.

Applying Parseval's Goldstein theorem[4] for H-transfoam to the relation (3·1) and (3·3) we get

$$\int_0^\infty t^{2\lambda-\frac{1}{2}}\, W_{k,m}\,(2\,\alpha\, t^2)\,\psi\,(t)\, dt = \frac{2^{-3\lambda-3v/2-2}\ \pi^{-\frac{1}{2}}\ \Gamma(v/2 + \lambda \pm \mu + 2)}{\alpha^{\lambda+v/2+1}\ \Gamma(v+3/2)\ \Gamma(5/2+\lambda - k - v/2)} \times$$

$$\times \int_0^\infty t^{v+\frac{1}{2}}\, g(t, \alpha)\ {}_3F_3\left[\begin{matrix}1,\, v/2 + \lambda \pm \mu + 2 \\ 3/2,\, v + 3/2,\, 5/2 + \lambda - k + v/2\end{matrix}\,;\, -\, t^2/8\alpha\right] dt.$$

Replacing t^2 by t in the left integral, multiplying both sides by $\alpha\,(2\alpha)^{-\frac{1}{4}}$ and on finally on replacing α by p we obtain the theorem.

Corollary.—In the theorem 2 if we take $k = \pm\,\mu = \frac{1}{4}$ we get

If

$$\phi(p) = L\,[t^{\lambda}\, \psi(t^{\frac{1}{2}})\,;\, p]$$

and

$$e^{\alpha p^2}\, \psi(p) = H\,[g\,(t, \alpha)\,;\, v]$$

then

$$\phi(p) = \frac{2^{-3\lambda-3v/2-9/4}\ \Gamma(\lambda + v/2 + 7/4)}{\pi^{\frac{1}{2}}\, p^{v/2+\lambda+\frac{3}{4}}\ \Gamma(v + 3/2)} \int_0^\infty t^{v+3/2}\ {}_2F_2\left[\begin{matrix}1,\, \lambda + v/2 + 7/4 \\ 3/2,\, v + 3/2\end{matrix}\,;\, t^2/8p\right] g(t, p)\, dt) \qquad \ldots.(3\cdot4)$$

provided $R(p) > o$, $R(2\lambda + v + 7/2) > o$, $\psi(p)$ is independent of α and the integral involved in (3·4) is absolutely convergent.

Example. If we take

$$\psi(p) = p^{-\frac{1}{2}} e^{-qp^2}$$

then since

$$g(t, \alpha) = \int_0^\infty e^{\alpha p^2} \psi(p) \, (pt)^{\frac{1}{2}} Y_v (pt) \, dp$$

we have (3, p. 106)

$$g(t, \alpha) = -\tfrac{1}{2} \left(\frac{\pi t}{q - \alpha}\right)^{\frac{1}{2}} e^{-t^2/8(q-\alpha)} \left[\tan \frac{v\pi}{2} I_{v/2} \left(t^2/8(q-\alpha)\right) + \frac{\text{Sec}\, v\pi/2}{\pi} K_{v/2} \left(t^2/8(q-\alpha)\right) \right]$$

where $R(\alpha) > o$ and $-1 < R(v) < 1$.

Also by hypothesis

$$\phi(p) = W[t^{\lambda-\frac{1}{2}} e^{-qt} ; K, \mu]$$

then [1, p. 13]

$$\phi(p) = \tfrac{1}{2} \frac{\Gamma(\lambda \pm \mu + 1)}{(2p)^{\lambda-\frac{1}{2}} \; \Gamma(\lambda - k + 3/2)} \, {}_2F_1 \left[\begin{matrix}\lambda \pm \mu + 1 \\ \lambda - k + 3/2\end{matrix} ; \tfrac{1}{2} - \frac{q}{2p}\right]$$

Using value of $g(t, p)$ in the relation (3·2) and equating this value of $\phi(p$ with the value of $\phi(p)$ obtained in the relation (2·1) we get

$$\int_0^\infty t^{v+2} e^{-t^2/8(q-p)} \, {}_3F_3 \left[\begin{matrix} 1, 2 + v/2 + \lambda \pm \mu \\ 3/2, v+3/2, 5/2+\lambda - k+v/2 \end{matrix} ; t^2/8p \right]$$

$$\left\{ \tan \frac{v\pi}{2} I_{v/2} \left(t^2/8(q-p)\right) + \frac{\text{Sec}\, v\pi/2}{\pi} K_{v/2} \left(t^2/8\,(q-p)\right) \right\} dt$$

$$= \frac{\Gamma(v + 3/2) \; \Gamma(5/2 + \lambda - k + v/2) \; \Gamma(\lambda \pm \mu + 1) \; (q-p)^{\frac{1}{2}} \, p^{v/2+\frac{1}{2}}}{\Gamma(\lambda + v/2 + 2 \pm \mu) \; \Gamma(\lambda - k + 3/2) \; 2^{5/2-2\lambda-3v/2}}$$

$${}_2F_1 \left[\begin{matrix}\lambda \pm \mu + 1 \\ \lambda - k + 3/2\end{matrix} ; \tfrac{1}{2} - \frac{q}{2p}\right] \qquad \ldots\ldots \quad (3·6)$$

where $R(p) > o$ and $-1 < R(v) < 1$.

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THERMO-ELASTIC DEFORMATION OF AN ISOTROPIC WIRE

By

MD. MAHFOOZ ALI SIDDIQUI

Department of Mathematics, Osmania University, Hyderabad–Dn.

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ABSTRACT

The case has been considered when a small increase of temperature of amount "T" is produced by heating the rod with a small cross-sectional area so that a uniform dilatation of amount proportional to "T" takes place without any corresponding change of pressure, the constant of proportionality being the coefficient of expansion, "C".

Under the above circumstances the stressstrain law enumerated by J. M. C. Dahamal and F. Neuman is :

$$T = (\lambda I_1 - \nu T_{ji}) S_{ji} + 2 \mu e_{ji}$$

where $\nu = K C$ (Bull madules $\times$ coeifficient of expension)

A natural modification of the above law is made in the present paper by using the following non-linear law with finite components of strain ε_{ij} :

$$T_{ij} = [\lambda I_1 (1 - I_1) - \nu T] S_{ij} + 2 [\mu - (\lambda + \mu) I_1] \varepsilon_{ij} - 4 \mu \varepsilon_{\alpha i} \varepsilon_{\alpha j}$$

As is to be expected the solution in not unique. The first order approximation of a solution shows that all linear elements are extended at the rate $CT/3$ which is in perfect agreement with the linear theory."

We assume the displacement components as under :

$$u = x(1-p)\,;\ v = y(1-q)\,;\ w = z(1-r) \qquad \ldots\ (1{\cdot}1)$$

where p, q, r, are real constants

The components of strain are :

$$\varepsilon_{xx} = \tfrac{1}{2}(2-p^2)\,;\ \varepsilon_{yy}\ \tfrac{1}{2}(1-q^2)\,;\ \varepsilon_{zz} = \tfrac{1}{2}(1-r^2)$$
$$\varepsilon_{y3} = \varepsilon_{3x} = \varepsilon_{xy} = 0 \qquad \ldots\ (1{\cdot}2)$$

First we adopt the linear law of Duhamal and Neumann but use the finite strain components leading to the stresses given by

$$X_x = \frac{\lambda}{2}(3 - p^2 - q^2 - r^2) + \mu(1 - p^2) - \nu T$$

$$Y_y = \frac{\lambda}{2}(3 - p^2 - q^2 - r^2) + \mu(1 - q^2) - \nu T \qquad \ldots\ (1{\cdot}3)$$

$$Z_z = \frac{\lambda}{2}(3 - p^2 - q^2 - r^2) + \mu(1 - r^2) - \nu T$$

$$Y_z = Z_x = X_y = 0$$

Since the curved boundary is stress free we have

$$X_x = Y_y = Z_z = 0 \qquad \ldots\ (1{\cdot}4)$$

so that it follows immediately that

$$p^2 = q^2 = r^2$$

Relations (1·3) now lead to

$$(1\ r^2)(\lambda + \tfrac{2}{3}\mu) = \tfrac{2}{3}\nu T \text{ giving } \lambda = (1 = \tfrac{2}{3} c T)^{\frac{1}{2}} \qquad \ldots . (1\cdot5)$$

Now we proceed to use the modified non-linear stress-strain law given above. The stresses are

$$X_x = \lambda\left[\tfrac{3}{2} - \tfrac{1}{2}(p^2+q^2+r^2)\right) - \lambda\left[\tfrac{9}{4} + \tfrac{1}{4}(p^2+q^2+r^2)^2 - \tfrac{3}{2}(p^2+q^2+r^2)\right]$$
$$+ \mu(1-p^2) - (\lambda+\mu)\left[\tfrac{3}{2} - \tfrac{1}{2}(p^2\ q^2\ r^2)\right](1-p^2) - \mu(1-p^2)^2 - \nu T$$
$$Y_y = \lambda\left[\tfrac{3}{2} - \tfrac{1}{2}(p^2+q^2+r^2)\right] - \lambda\left[\tfrac{9}{4} + \tfrac{1}{4}(p^2+q^2+r^2)^2 - \tfrac{3}{2}(p^2+q^2+r^2)\right]$$
$$+ \mu(1-q^2) - (\lambda+\mu)\left[\tfrac{3}{2} - \tfrac{1}{2}(p^2+q^2+r^2)^2 - \nu T\right.$$
$$Z_z = \lambda\left[\tfrac{3}{2} - \tfrac{1}{2}(p^2+q^2+r^2)\right] - \lambda\left[\tfrac{9}{4} + \tfrac{1}{4}(p^2+q^2+r^2)^2 - \tfrac{3}{2}(p^2+q^2+r^2)\right]$$
$$+ \mu(1-r^2) - (\lambda+\mu)\left[\tfrac{3}{2} - \tfrac{1}{2}(p^2+q^2+r^2)\right](1-r^2) - \mu(1-r^2)^2 - \nu T$$
$$Y_z = Y_x = X_y = 0$$

The body-stress equations of equilibrium are identically satisfied, and the curved boundary being stress-free it follows that $p^2 = q^2 = r^2$; and we are lead to the following biquadratic equation in r :

$$(1-r^2)\{1 - \tfrac{5}{2}(1-r^2)\} = \frac{2\nu T}{3\lambda + 2\mu} \qquad \ldots . (1\cdot7)$$

since $\nu = (\lambda + \frac{2}{3}\mu)\,c$ where $k = (\lambda + \frac{2}{3}\mu)$, we have

$$\nu = \left\{\tfrac{4}{5} \pm \tfrac{1}{5}\sqrt{1 - \frac{20}{3} C T}\right\}^{\frac{1}{2}} \qquad \ldots . (1\cdot8)$$

Obviously one solution of (1·8) is

$$\nu = 1 - \frac{C T}{3}$$

where higher powers of CT have been neglected because both C and T are small quantities.

Corresponding to this approximate solution the extension per unit length becomes

$$e = \frac{(1-\nu)}{\nu z} = \frac{CT}{3}\left(1 - \frac{CT}{3}\right)^{-1}$$
$$= \frac{CT}{3}$$

This is in perfect arrangement with the ordinary result.

Incidently, the calculations reveal that the situation is the same as corresponds to the case of a hydra-static pressure with a change of sign.

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ON GENERALISED BESSEL FUNCTIONS AND THEIR ASSOCIATED FUNCTIONS

By

S. L. GUPTA*

Department of Mathematics, University of Jodhpur, Jodhpur

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ABSTRACT

In this paper a few functions associated to the generalised Bessel functions have been defined and certain recurrence formulae and expansions have been obtained. Associated functions of the first and the second type have also been expressed in terms of infinite series of $J^{(1)}_m\{x_n\}$'s and $J^{(2)}_m\{x_n\}$'s respectively.

1. Generalised Bessel functions of the first kind.

Taking $J^{(1)}_\mu\{x_n\}$ and $J^{(2)}_\mu\{x_n\}$ the first and the second type of the Bessel functions of the first kind and of arguments $(x_1, \ldots, x_n)$ and unrestricted order μ respectively defined by the equations :

$$J^{(1)}_\mu\{x_n\} = \sum^\infty J_{t_1}(x_1) \ldots J_{t_n}(x_n), \tag{1}$$

$t_1, \ldots, t_n = -\infty$ except t_r, which is given by $t_1 + \ldots + nt_n = \mu$

where r may be given any integral value from 1 to n,

and

$$J^{(2)}_\mu\{x_n\} = \sum^\infty J_{t_1}(x_1)\, I_{t_2}(x_2)\, J_{t_3}(x_3) \ldots \tag{2}$$

$t_1, \ldots, t_n = -\infty$ except t_r, which is given by $t_1 + \ldots + nt_n = \mu$,

where the last term within the summation is $J_{t_n}(x_n)$ or $I_{t_n}(x_n)$ according as n is odd or even,

we can show that

$$\mu J^{(1)}_\mu\{x_n\} = \tfrac{1}{2}x_1\,(J^{(1)}_{\mu-1}\{x_n\} + J^{(1)}_{\mu+1}\{x_n\}) + \ldots + \tfrac{1}{2}n\,x_n\,(J^{(1)}_{\mu-n}\{x_n\} + J^{(1)}_{\mu+n}\{x_n\}), \tag{3}$$

$$\frac{\partial J^{(1)}_\mu\{x_n\}}{\partial x_p} = \tfrac{1}{2}\,(J^{(1)}_{\mu-p}\{x_n\} - J^{(1)}_{\mu+p}\{x_n\}), \tag{4}$$

$$\mu J^{(2)}_\mu\{x_n\} = \tfrac{1}{2}x_1\,(J^{(2)}_{\mu-1}\{x_n\} + J^{(2)}_{\mu+1}\{x_n\}) + \ldots + \tfrac{1}{2}n\,x_n\,(J^{(2)}_{\mu-n}\{x_n\} + (-1)^{n+1}J^{(2)}_{\mu+n}\{x_n\}), \tag{5}$$

Present address : Department of Mathematics, University of Roorkee, Roorkee.

$$\frac{\partial J^{(2)}_{\mu}\{x_n\}}{\partial x_p} = \tfrac{1}{2}\left(J^{(2)}_{\mu-p}\{x_n\} + (-1)^p J^{(2)}_{\mu+p}\{x_n\}\right) \tag{6}$$

$$s^{\mu} J^{(1)}_{\mu}\left\{x_n\left(s^n + \frac{1}{s^n}\right)\right\} = \sum_{m=-\infty}^{\infty} s^{2m} J^{(1)}_{\mu-m}\{x_n\} J^{(1)}_{m}\{x_n\}, \tag{7}$$

$$s^{\mu} J^{(2)}_{\mu}\left\{x_n\left(s^n + \frac{(-1)^{n+1}}{s^n}\right)\right\} = \sum_{m=-\infty}^{\infty} s^{2m} J^{(1)}_{\mu-m}\{x_n\} J^{(1)}_{m}\{(-1)^{n+1} x_n\}, \tag{8}$$

where $|s| < |$ in case $\mu \neq 0, \pm 1, \pm 2, \ldots.$

To prove (3) from (1), we find that

$$\tfrac{1}{2}px_p\left(J^{(1)}_{\mu-p}\{x_n\} + J^{(1)}_{\mu+p}\{x_n\}\right) = \tfrac{1}{2}\sum^{\infty} px_p J_{t_1}(x_1)\ldots J_{t_n}(x_n)$$

$t_1, \ldots, t_n = -\infty$ except t_r, which is given by $t_1 + \ldots + nt_n = \mu - p$

$$+\tfrac{1}{2}\sum^{\infty} px_p J_{t_1}(x_1)\ldots J_{t_n}(x_n)$$

$t_1, \ldots, t_n = -\infty$ except t_r, which is given by $t_1 + \ldots + nt_n = \mu + p$

$$= \tfrac{1}{2}\sum^{\infty} px_p\left(J_{t_p-1}(x_p) + J_{t_p+1}(x_p)\right) J_{t_1}(x_1)\ldots J^{*}_{t_n}(x_n)$$

$t_1, \ldots, t_n = -\infty$ except t_r, which is given by $t_1 + \ldots + nt^*_n = \mu$

The star in t^*_n indicates that the term corresponding to t_p does not occur within the space marked by .... On using the recurrence relation

$$tJ_t(x) = \tfrac{1}{2}x\left(J_{t-1}(x) + J_{t+1}(x)\right),$$

we get

$$\tfrac{1}{2}p\,x_p\left(J^{(1)}_{\mu-p}\{x_n\} + J^{(1)}_{\mu+p}\{x_n\}\right) = \sum^{\infty} pt_p J_{t_1}(x_1)\ldots J_{t_n}(x_n)$$

$t_1, \ldots, t_n = -\infty$ except t_r, which is given by $t_1 + \ldots + nt_n = \mu$

Thus

$$\tfrac{1}{2}x_1\left(J^{(1)}_{\mu-1}\{x_n\} + J^{(1)}_{\mu+1}\{x_n\}\right) + \ldots \tfrac{1}{2}n\,x_n\left(J^{(1)}_{\mu-n}\{x_n\} + J^{(1)}_{\mu+n}\{x_n\}\right)$$

$$= \sum^{\infty}(t_1 + \ldots nt_n) J_{t_1}(x_1)\ldots J_{t_n}(x_n)$$

$t_1, \ldots, t_n = -\infty$ except t_r, which is given by $t_1 + \ldots + nt_n = \mu$

$$= \mu J^{(1)}_{\mu}\{x_n\}.$$

Similarly (4)-(6) can be proved.

To prove (7) we use [1, p. 102 (39)]

$$J_t\left(x\left(s+\frac{1}{s}\right)\right)=\sum_{k=-\infty}^{\infty} s^{2k-t}\, J_{t-k}\,(x)\, J_k\,(x) \tag{a}$$

where $|s|<|$ in case $t\neq 0, \pm 1, \pm 2, \ldots.$

so that

$$J_{\mu}^{(1)}\left\{x_n\left(s^n+\frac{1}{s^n}\right)\right\}=\sum^{\infty} s^{2(k_1+\ \ldots\ +nk_n)-(t_1+\ \ldots\ +nt_n)}$$
$$\times\ J_{t_1-k_1}\,(x)\ldots J_{t_n-k_n}\,(x)\, J_{k_1}\,(x)\ldots J_{k_n}\,(x)$$

$t_1, \ldots, t_n\,;\, k_1, \ldots, k_n=-\infty$ except t_r which is given by $t_1+\ \ldots\ n_{t_n}=\mu$

where $|s|<|$ in case $t\neq 0, \pm 1, \pm 2, \ldots$

Putting $k_1+\ \ldots\ +nk_n=m$, where $m=0, \pm 1, \pm 2, \ldots$ we have

$$J_{\mu}^{(1)}\left\{x_n\left(s^n+\frac{1}{s^n}\right)\right\}=\sum^{\infty} s^{2m-\mu} J_{t_1-k_1}\,(x)\ldots J_{t_n-k_n}\,(x_n)\, J_{k_1}\,(x_1)\ldots J_{k_n}\,(x_n)$$

$t_1\ldots, t_n\,;\, k_1\ldots, k_n\,;\, m=-\infty$ except t_r, which is given $t_1+\ \ldots\ +n_{t_n}=\mu$

and $k_1+\ \ldots\ +nk_n=m$

Again if we put $t_1-k_1=q_1, \ldots, t_n-k_n=q_n$ we get

$$s^{\mu} J_{\mu}^{(1)}\left\{x_n\left(s^n+\frac{1}{s^n}\right)\right\}=\sum^{\infty} s^{2m}\, J_{q_1}\,(x_1)\ldots J_{q_n}\,(x_n) J_{k_1}\,(x_1)\ldots J_{k_n}\,(x_n)$$

$q_1\ldots, q_n\,;\, k_1\ldots, k_n\,;\, m=-\infty$ except q_r which is given by $q_1+\ \ldots\ +nq_n$
$=\mu-m$ and $k_1+\ \ldots\ +nk_n=m$

$$=\sum^{\infty} s^{2m}\, J_{\mu-m}^{(1)}\{x_n\} J_{k_1}\,(x_1)\ldots J_{k_n}\,(x_n)$$

$k_1\ldots, k_n\,;\, m=-\infty$ provided $k_1+\ \ldots\ +nk_n=m$

$$=\sum_{m=-\infty}^{\infty} s^{2m}\, J_{\mu-m}^{(1)}\,\{x_n\}\, J_m^{(1)}\,\{x_n\},$$

where $|s|<|$ in case $\mu\neq 0, \pm 1, \pm 2, 0\ldots$

In a similar manner (8) can also be easily proved by using (a) and [1, p. 102—(40)].

$$I_t\left(x\left(\frac{1}{s}-s\right)\right)=\sum_{k=-\infty}^{\infty} s^{-2k-t}\, J_{t+k}\,(x)\, J_k\,(x)$$

where $|s|<|$ in case $t\neq 0, \pm 1, \pm 2, \ldots$

2. First kind of the generalised modified Bessel functions.

The first and the second type of the generalised modified Bessel functions of the first kind which are denoted by $I^{(1)}_{\mu}\{x_n\}$ and $I^{(2)}_{\mu}\{x_n\}$ respectively are de-defined by the equations

$$I^{(1)}_{\mu}\{x_n\} = \sum^{\infty} I_{t_1}(x_1) \dots I_{t_n}(x_n) \tag{1}$$

$$t_1, \dots, t_n = -\infty \text{ except } t_r \text{ which is given by } t_1 + \dots + nt_n = \mu$$

and

$$I^{(2)}_{\mu}\{x_n\} = \sum^{\infty} I_{t_1}(x_1) J_{t_2}(x_2) J_{t_3}(x_3) \tag{2}$$

$$t_1 \dots t_n = -\infty \text{ except } t_r, \text{ which is given by } t_1 + \dots + nt_n = \mu$$

where the last term within the summation is $I_{t_n}(x_n)$ or $J_{t_n}(x_n)$ according as n is odd or even.

From these definitions it can be easily shown that

$$\mu I^{(1)}_{\mu}\{x_n\} = \tfrac{1}{2}x_1\left(I^{(1)}_{\mu-1}\{x_n\} - I^{(1)}_{\mu+1}\{x_n\} + \dots\right.$$
$$\left. + \tfrac{1}{2}n\, x_n\left(I^{(1)}_{\mu-n}\{x_n\} - I^{(1)}_{\mu+n}\{x_n\}\right)\right), \tag{3}$$

$$\frac{\partial I^{(1)}_{\mu}\{x_n\}}{\partial x_p} = \tfrac{1}{2}\left(I^{(1)}_{\mu-p}\{x_n\} + I^{(1)}_{\mu+p}\{x_n\}\right) \tag{4}$$

$$\mu\, I^{(2)}_{\mu}\{x_n\} = \tfrac{1}{2}x_1\left(I^{(2)}_{\mu-1}\{x_n\} - I^{(2)}_{\mu+1}\{x_n\}\right) + \dots +$$
$$+ \tfrac{1}{2}n\, x_n\left(I^{(2)}_{\mu-n}\{x_n\} + (-1)^n I^{(2)}_{\mu+n}\{x_n\}\right), \tag{5}$$

$$\frac{\partial I^{(2)}_{\mu}\{x_n\}}{\partial x_p} = \tfrac{1}{2}\left(I^{(2)}_{\mu-p}\{x_n\} + (-1)^{p+1} I^{(2)}_{\mu}\{x_n\}\right); \tag{6}$$

and

$$s^{\mu} I^{(1)}_{\mu}\left\{x_n\left(\frac{1}{s^n} - s^n\right)\right\} = \sum_{m=-\infty}^{\infty} s^{-2m} J^{(1)}_{\mu-m}\{x_n\}\, J^{(1)}_{m}\{x_n\} \tag{7}$$

$$s^{\mu} I^{(2)}_{\mu}\left\{x_n\left(\frac{1}{s^n} + (-1)^{n+1} s^n\right)\right\} = \sum_{m=-\infty}^{\infty} s^{-2m} J^{(1)}_{\mu+m}\{x_n\}\, J^{(1)}_{m}\{(-1)^{n+1} x_n\}, \tag{8}$$

where $|s| < |$ in case $\mu \neq 0, \pm 1, \pm 2, \dots$

3. The functions $E^{(1)}_{m}\{x_n\}$ and $E^{(2)}_{m}\{x_n\}$.

If μ takes integral values then from § 1 (1, 2) it can be shown that

$$J^{(1)}_{m}\{x_n\} = \frac{1}{\pi}\int_0^{\pi} \cos(m\theta - x_1 \sin\theta - \dots - x_n \sin n\theta)\, d\theta \tag{1}$$

and

$$J^{(2)}_m\{x_n\} = \frac{1}{\pi}\int_0^\pi e^{x_2 \cos 2\theta + x_4 \cos 4\theta + \dots}$$

$$\times \cos(m\theta - x_1 \sin\theta - x_3 \sin 3\theta - \dots)\, d\theta. \qquad (2)$$

We can now define the functions $E^{(1)}_m\{x_n\}$ and $E^{(2)}_m\{x_n\}$, which are generalisation of the function $E_m(x)$ associated to $J_m(x)$, by the equations

$$E^{(1)}_m\{x_n\} = \frac{1}{\pi}\int_0^\pi \sin(m\theta - x_1\sin\theta - \dots - x_n \sin n\theta)\, d\theta, \qquad (3)$$

and

$$E^{(2)}_m\{x_n\} = \frac{1}{\pi}\int_0^\pi e^{x_2 \cos 2\theta + x_4 \cos 4\theta + \dots}$$

$$\times \sin(m\theta - x_1 \sin\theta - x_3 \sin 3\theta - \dots)\, d\theta. \qquad (4)$$

It can be easily shown that

$$E^{(1)}_m\{x_n\} = (-1)^{m+1}\, E^{(1)}_m\{(-1)^n x_n\},\; E^{(1)}_{-m}\{-x_n\} = -E^{(1)}_m\{x_n\}; \qquad (5)$$

$$E^{(2)}_m\{x_n\} = (-1)^{m+1}\, E^{(2)}_m\{(-1)^n x_n\},\; E^{(2)}_{-m}\{x_n\} = -E^{(2)}_m\{(-1)^{n+1} x_n\}; \qquad (6)$$

$$J^{(1)}_m\{x_n\} = (-1)^m\, J^{(1)}_m\{(-1)^n x_n\},\; J^{(1)}_{-m}\{x_n\} = (-1)^m\, J^{(1)}_M\{(-1)^{n+1} x_n\}; \qquad (7)$$

$$J^{(2)}_m\{x_n\} = (-1)^m\, J^{(2)}_m\{(-1)^n x_n\},\; J^{(2)}_{-m}\{x_n\} = (-1)^m\, J^{(2)}_M\{x_n\}, \qquad (8)$$

$$mE^{(1)}_m\{x_n\} - \frac{1}{\pi}(1-(-1)^m) = \tfrac{1}{2}x_1(E^{(1)}_{m-1}\{x_n\} + E^{(1)}_{m+1}\{x_n\}) + \dots$$

$$+ \tfrac{1}{2}n x_n (E^{(1)}_{M-n}\{x_n\} + E^{(1)}_{M+n}\{x_n\}), \qquad (9)$$

$$\frac{\partial E^{(1)}_m\{x_n\}}{\partial x_p} = \tfrac{1}{2}(E^{(1)}_{m-p}\{x_n\} - E^{(1)}_{m+p}\{x_n\}); \qquad (10)$$

$$m E^{(2)}_m\{x_n\} - \frac{1}{\pi} e^{x_2+x_4+\dots}(1+(-1)^m) = \tfrac{1}{2}x_1(E^{(2)}_{m-1}\{x_n\} + E^{(2)}_{m+1}\{x_n\} +$$

$$\dots + \tfrac{1}{2}n x_n (E^{(2)}_{m-n}\{x_n\} + (-1)^{n+1} E^{(2)}_{m+n}\{x_n\}), \qquad (11)$$

$$\frac{\partial E^{(2)}_m\{x_n\}}{\partial x_p} = \tfrac{1}{2}(E^{(2)}_{m-p}\{x_n\} + (-1)^p E^{(2)}_{m+p}\{x_n\}); \qquad (12)$$

$$E^{(1)}_m\{x_n\} = \frac{1}{\pi}\cdot\sum_{k=-\infty \text{ except } k=\pm m}^{\infty} J^{(2)}_k\{x_n\} \times \frac{(1-(-1)^{m-k})}{m-k}, \qquad (13)$$

$$E_m^{(2)}\{x_n\} = \frac{1}{\pi}\sum_{\substack{k=-\infty \\ \text{except } k=\pm m}}^{\infty} J_k^{(2)}\{x_n\}\frac{(1-(-1)^{m-k})}{m-k}. \tag{14}$$

From the definitions of $E_m^{(1)}\{x_n\}$ and $E_m^{(2)}\{x_n\}$ and (5)—(12) it evidently appears that these functions may satisfy numerous results analogous to those which are satisfied by $J_m^{(1)}\{x_n\}$ and $J_m^{(2)}\{x_n\}$ respectively.

4. The functions $\mathbf{J}_\mu^{(1)}\{x_n\}$, $\mathbf{J}_\mu^{(2)}\{x_n\}$, $\mathbf{E}_\mu^{(1)}\{x_n\}$ and $\mathbf{E}_\mu^{(2)}\{x_n\}$.

We now extend to n arguments the definitions of the functions $\mathbf{J}_\mu(x)$ and $\mathbf{E}_\mu(x)$ which were originally studied by Anger and H. F. Weber.

The functions $\mathbf{J}_\mu^{(1)}\{x_n\}$ and $\mathbf{J}_\mu^{(2)}\{x_n\}$ are the generalisations of the functions $\mathbf{J}_\mu(x)$ and they are defined by the equations

$$\mathbf{J}_\mu^{(1)}\{x_n\} = \frac{1}{\pi}\int_0^\pi \cos(\mu\theta - x_1\sin\theta - \ldots - x_n\sin n\theta)\,d\theta, \tag{1}$$

$$\mathbf{J}_\mu^{(2)}\{x_n\} = \frac{1}{\pi}\int_0^\pi e^{x_2\cos 2\theta + x_4\cos 4\theta + \ldots} \times \cos(\mu\theta - x_1\sin\theta - \ldots - x_n\sin n\theta)\,d\theta \tag{2}$$

Obviously, these functions reduce to $J_m^{(1)}\{x_n\}$ and $J_m^{(2)}\{x_n\}$ respectively when μ takes the integral value m. Also it can be easily seen that when μ is not an integer, the functions $\mathbf{J}_\mu^{(1)}\{x_n\}$, $J_\mu^{(1)}\{x_n\}$ and $\mathbf{J}_\mu^{(2)}\{x_n\}$, $J_\mu^{(2)}\{x_n\}$ are distinct.

Similarly in view of the functions $E_m^{(1)}\{x_n\}$ and $E_m^{(2)}\{x_n\}$ the functions $\mathbf{E}_\mu^{(1)}\{x_n\}$ and $\mathbf{E}_\mu^{(2)}\{x_n\}$ which are the generalisations of $\mathbf{E}_\mu(x)$ are defined by the equations

$$\mathbf{E}_\mu^{(1)}\{x_n\} = \frac{1}{\pi}\int_0^\pi \sin(\mu\theta - x_1\sin\theta - \ldots - x_n\sin n\theta)\,d\theta, \tag{3}$$

$$\mathbf{E}_\mu^{(2)}\{x_n\} = \frac{1}{\pi}\int_0^\pi e^{x_2\cos 2\theta + x_4\cos 4\theta + \ldots} \times \sin(\mu\theta - x_1\sin\theta - x_3\sin 3\theta - \ldots)\,d\theta. \tag{4}$$

From the above definitions we can show that

$$\left.\begin{aligned}
\mathbf{J}^{(1)}_{\mu}\{x_n\} &= \frac{2}{\pi}\int_0^{\pi/2}\cos(\mu\theta - x_1\cos\theta + x_3\cos 3\theta - \ldots) \\
&\qquad \times \cos(\mu\theta + x_2\sin 2\theta - x_4\sin 4\theta + \ldots)\,d\theta \\
\mathbf{J}^{(2)}_{\mu}\{x_n\} &= \frac{2}{\pi}\int_0^{\pi/2} e^{x_2\cos 2\theta + x_4\cos 4\theta + \ldots} \\
&\qquad \times \cos\mu\theta\cos\left(\frac{\pi}{2}\mu - x_1\cos\theta + x_3\cos 3\theta - \ldots\right)d\theta, \\
\mathbf{E}^{(1)}_{\mu}\{x_n\} &= \frac{2}{\pi}\int_0^{\pi/2}\sin\left(\frac{\pi}{2}\mu - x_1\cos\theta + x_3\cos 3\theta - \ldots\right) \\
&\qquad \times \cos(\mu\theta + x_2\sin 2\theta - x_4\sin 4\theta + \ldots)\,d\theta, \\
\mathbf{E}^{(2)}_{\mu}\{x_n\} &= \frac{2}{\pi}\int_0^{\pi/2} e^{-x_2\cos 2\theta + x_4\cos 4\theta - \ldots} \\
&\qquad \cos\mu\theta\sin\left(\frac{\pi}{2}\mu - x_1\cos\theta + x_3\cos 3\theta - \ldots\right)d\theta
\end{aligned}\right\} \tag{5}$$

$$\mathbf{J}^{(1)}_{\mu}\{x_n\} = \mathbf{J}^{(1)}_{-\mu}\{-x_n\},\ \mathbf{J}^{(2)}_{-\mu}\{x_n\} = \mathbf{J}^{(2)}_{\mu}\{(-1)^n x_n\}; \tag{6}$$

$$\mathbf{E}^{(1)}_{\mu}\{x_n\} = -\mathbf{E}^{(1)}_{-\mu}\{-x_n\},\ \mathbf{E}^{(2)}_{-\mu}\{x_n\} = -\mathbf{E}^{(2)}_{\mu}\{(-1)^n x_n\}; \tag{7}$$

$$\left.\begin{aligned}
\mu\mathbf{J}^{(1)}_{\mu}\{x_n\} - \frac{1}{\pi}\sin\mu\pi &= \tfrac{1}{2}x\left(\mathbf{J}^{(1)}_{\mu-1}\{x_n\} + \mathbf{J}^{(1)}_{\mu+1}\{x_n\}\right) + \ldots \\
&\quad + \tfrac{1}{2}n\,x_n\left(\mathbf{J}^{(1)}_{\mu-n}\{x_n\} + \mathbf{J}^{(1)}_{\mu+n}\{x_n\}\right), \\
\mu\mathbf{J}^{(2)}_{\mu}\{x_n\} - \frac{1}{\pi}e^{x_2+x_4+\ldots}\sin\mu\pi &= \tfrac{1}{2}x_1\left(\mathbf{J}^{(2)}_{\mu-1}\{x_n\} + \mathbf{J}^{(2)}_{\mu+1}\{x_n\}\right) + \ldots \\
&\quad + \tfrac{1}{2}n\,x_n\left(\mathbf{J}^{(2)}_{\mu-n}\{x_n\} + (-1)^{n+1}\mathbf{J}^{(2)}_{\mu+n}\{x_n\}\right. \\
\mu\mathbf{E}^{(1)}_{\mu}\{x_n\} - \frac{2}{\pi}\sin^2\frac{\mu\pi}{2} &= \tfrac{1}{2}x_1\left(\mathbf{E}^{(1)}_{\mu+1}\{x_n\} + \mathbf{E}^{(1)}_{\mu-1}\{x_n\}\right) + \ldots \\
&\quad + \tfrac{1}{2}n x_n\left(\mathbf{E}^{(1)}_{\mu-n}\{x_n\} + \mathbf{E}^{(1)}_{\mu+n}\{x_n\}\right), \\
\mu\mathbf{E}^{(2)}_{\mu}\{x_n\} - \frac{2}{\pi}e^{x_2+x_4+\ldots}\sin^2\frac{\mu\pi}{2} &= \tfrac{1}{2}x_1\left(\mathbf{E}^{(2)}_{\mu-1}\{x_n\}\right. \\
&\quad \left. + \mathbf{E}^{(2)}_{\mu+1}\{x_n\}\right) + \ldots + \tfrac{1}{2}n x_n\left(\mathbf{E}^{(2)}_{\mu-n}\{x_n\} + (-1)\,\mathbf{E}^{(2)}_{\mu+n}\{x_n\}\right)
\end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned}
\frac{\partial\mathbf{J}^{(1)}_{\mu}\{x_n\}}{\partial x_p} &= \tfrac{1}{2}\left(\mathbf{J}^{(1)}_{\mu-p}\{x_n\} - \mathbf{J}^{(1)}_{\mu+p}\{x_n\}\right), \\
\frac{\partial\mathbf{J}^{(2)}_{\mu}\{x_n\}}{\partial x_p} &= \tfrac{1}{2}\left(\mathbf{J}^{(2)}_{\mu-p}\{x_n\} + (-1)^p\mathbf{J}^{(2)}_{\mu+p}\{x_n\}\right), \\
\frac{\partial\mathbf{E}^{(1)}_{\mu}\{x_n\}}{\partial x_p} &= \tfrac{1}{2}\left(\mathbf{E}^{(1)}_{\mu-p}\{x_n\} - \mathbf{E}^{(1)}_{\mu+p}\{x_n\}\right), \\
\frac{\partial\mathbf{E}^{(2)}_{\mu}\{x_n\}}{\partial x_p} &= \tfrac{1}{2}\left(\mathbf{E}^{(2)}_{\mu-p}\{x_n\} + (-1)^p\mathbf{E}^{(2)}_{\mu+p}\{x_n\}\right);
\end{aligned}\right\} \tag{9}$$

$$
\left.
\begin{aligned}
\mathbf{J}^{(1)}_{\mu}\{x_n\} &= \frac{1}{\pi}\sum_{m=-\infty}^{\infty}(-1)^m J^{(1)}_{m}\{x_n\}\frac{\sin\mu\pi}{\mu-m} = \frac{\sin\mu\pi}{\pi} \\
&\quad\times\left(\frac{J^{(1)}_0\{x_n\}}{\mu} + \sum_{m=1}^{\infty}(-1)^m\left(\frac{J^{(1)}_{-m}\{x_n\}}{\mu+m} + \frac{J^{(1)}_{m}\{x_n\}}{\mu-m}\right)\right) \\
\mathbf{J}^{(2)}_{\mu}\{x_n\} &= \frac{1}{\pi}\sum_{m=-\infty}^{\infty}(-1)^m J^{(2)}_{m}\{x_n\}\frac{\sin\mu\pi}{\mu-m} \\
&= \frac{\sin\mu\pi}{\pi}\left(\frac{J^{(2)}_0\{x_n\}}{\mu} + \sum_{m=1}^{\infty}\left(\frac{1}{\mu+m} + \frac{(-1)^m}{\mu-m}\right)J^{(2)}_{m}\{x_n\}\right) \\
\mathbf{E}^{(1)}_{\mu}\{x_n\} &= \frac{1}{\pi}\sum_{m=-\infty}^{\infty} J^{(1)}_{m}\{x_n\}\frac{1-(-1)^m\cos\pi\mu}{\mu-m} \\
&= \frac{2}{\pi}\left(\frac{\sin^2\frac{1}{2}\mu\pi}{\mu}J^{(1)}_0\{x_n\} + \sum_{m=1}^{\infty}\sin\tfrac{1}{2}(\mu+m)\pi\left(\frac{J^{(1)}_{m}\{x_n\}}{\mu+m} + \frac{J^{(1)}_{m}\{x_n\}}{\mu-m}\right)\right) \\
\mathbf{E}^{(2)}_{\mu}\{x_n\} &= \frac{1}{\pi}\sum_{m=-\infty}^{\infty} J^{(2)}_{m}\{x_n\}\frac{1-(-1)^m\cos\mu\pi}{\mu-m} \\
&= \frac{2}{\pi}\left(\frac{\sin^2\frac{1}{2}\mu\pi}{\mu}J^{(2)}_0\{x_n\} + \sum_{m=1}^{\infty}\sin\tfrac{1}{2}(\mu+m)\pi\left(\frac{(-1)^m}{\mu+m} + \frac{1}{\mu-m}\right)J^{(2)}_{m}\{x_n\}\right)
\end{aligned}
\right\}\quad(10)
$$

$$
\left.
\begin{aligned}
&\tfrac{1}{2}\pi\int_0^{2\pi}\cos(\mu\theta - x_1\sin\theta - \ldots - x_n\sin n\theta)\,d\theta = \tfrac{1}{2}(\mathbf{J}^{(1)}_{\mu}\{x_n\} + \\
&\qquad \cos\mu\pi\,\mathbf{J}^{(1)}_{\mu}\{(-1)^n x_n\}) - \tfrac{1}{2}\sin\pi\mu\,\mathbf{E}^{(1)}_{\mu}\{(-1)^n x_n\} \\
&\tfrac{1}{2}\pi\int_0^{2\pi}\sin(\mu\theta - x_1\sin\theta - \ldots - x_n\sin n\theta)\,d\theta = \tfrac{1}{2}(\mathbf{E}^{(1)}_{\mu}\{x_n\} + \\
&\qquad \cos\pi\mu\,\mathbf{E}^{(1)}_{\mu}\{(-1)^n x_n\}) + \tfrac{1}{2}\sin\mu\pi\,\mathbf{J}^{(1)}_{\mu}\{(-1)^n x_n\} \\
&\tfrac{1}{2}\pi\int_0^{2\pi} e^{x_2\cos 2\theta + x_4\cos 4\theta + \ldots}\cos(\mu\theta - x_1\sin\theta - \\
&\qquad x_3\sin 3\theta - \ldots)\,d\theta = \cos\mu\pi\,\mathbf{J}^{(2)}_{\mu}\{x_n\} + \cos\mu\pi\,\sin\mu\pi\,\mathbf{E}^{(2)}_{\mu}\{x_n\} \\
&\tfrac{1}{2}\pi\int_0^{2\pi} e^{x_2\cos 2\theta + x_4\cos 4\theta + \ldots}\sin(\mu\theta - x_1\sin\theta - \\
&\qquad x_3\sin 3\theta - \ldots)\,d\theta = \cos\mu\pi\,\mathbf{J}^{(2)}_{\mu}\{x_n\} + \sin^2\mu\pi\,\mathbf{E}^{(2)}_{\mu}\{x_n\}
\end{aligned}
\right\}\quad(11)
$$

On replacing θ by $\pi-\theta$ in the integrands in (1) – (4), we obtain the following results :

$$\left.\begin{aligned}
\mathbf{J}_{\mu}^{(1)}\{x_n\} &= \cos\mu\pi\,\mathbf{J}_{\mu}^{(1)}\{(-1)^n x_n\} + \sin\mu\pi\,\mathbf{E}_{\mu}^{(1)}\{(-1)^n x_n\},\\
\mathbf{J}_{\mu}^{(2)}\{x_n\} &= \cos\mu\pi\,\mathbf{J}_{\mu}^{(2)}\{(-1)^n x_n\} + \sin\mu\pi\,\mathbf{E}_{\mu}^{(2)}\{(-1)^n x_n\},\\
\mathbf{E}_{\mu}^{(1)}\{x_n\} &= \sin\mu\pi\,\mathbf{J}_{\mu}^{(1)}\{(-1)^n x_n\} - \cos\mu\pi\,\mathbf{E}_{\mu}^{(1)}\{(-1)^n x_n\},\\
\mathbf{E}_{\mu}^{(2)}\{x_n\} &= \sin\mu\pi\,\mathbf{J}_{\mu}^{(2)}\{(-1)^n x_n\} - \cos\mu\pi\,\mathbf{E}_{\mu}^{(2)}\{(-1)^n x_n\},
\end{aligned}\right\} \tag{12}$$

5. The functions $T_m^{(1)}\{x_n\}$ and $T_m^{(2)}\{x_n\}$.

Among the functions belonging to the class $\Upsilon_m(x)$, the function $T_m(x)$ is most simply represented by the definite integral

$$T_m(x) = \frac{2}{\pi}\int_0^{\pi}\left(\theta-\frac{\pi}{2}\right)\sin(m\theta - x_1\sin\theta)\,d\theta.$$

Various properties of this function were studied by Schläfli [2, p. 341], also, from the form of the above integral itself an association of $T_m(x)$ with $J_m(x)$ is evident. The functions $T_m^{(1)}\{x_n\}$ and $T_m^{(2)}\{x_n\}$ are the generalisations of the function $T_m(x)$ and they are defined by the equations :

$$T_m^{(1)}\{x_n\} = \frac{2}{\pi}\int_0^{\pi}\left(\theta-\frac{\pi}{2}\right)\sin(m\theta - x_1\sin\theta - \ldots - x_n\sin n\theta)\,d\theta, \tag{1}$$

and

$$T_m^{(2)}\{x_n\} = \frac{2}{\pi}\int_0^{\pi}\left(\theta-\frac{\pi}{2}\right)e^{x_2\cos 2\theta + x_4\cos 4\theta + \ldots}$$
$$\times\sin(m\theta - x_1\sin\theta - x_3\sin 3\theta - \ldots)\,d\theta, \tag{2}$$

from these we can show that

$$T_m^{(1)}\{x_n\} = (-1)^m\,T_m^{(1)}\{(-1)^n x_n\},\quad T_{-m}^{(1)}\{x_n\} =$$
$$(-1)^{m+1}\,T_m^{(1)}\{(-1)^{n+1}x_n\}; \tag{3}$$

$$T_m^{(2)}\{x_n\} = (-1)^m\,T_m^{(2)}\{(-1)^n x_n\},\quad T_{-m}^{(2)}\{x_n\} = (-1)^{m+1}\,T_m^{(2)}\{x_n\}; \tag{4}$$

$$\left.\begin{aligned}
&mT^{(1)}_{m}\{x_n\} - 2J^{(1)}_{m}\{x_n\} + 2\cos^2 \tfrac{1}{2}m\pi = \tfrac{1}{2}x_1 \left(T^{(1)}_{m-1}\{x_n\}\right.\\
&\qquad + \left. T^{(1)}_{m+1}\{x_n\}\right) + \ldots + \tfrac{1}{2}nx_n \left(T^{(1)}_{m-n}\{x_n\} + T^{(1)}_{m+n}\{x_n\}\right),\\
&mT^{(2)}_{m}\{x_n\} - 2J^{(2)}_{m}\{x_n\} + 2\cos^2 \tfrac{1}{2}m\pi\, e^{x_2 + x_4 + \cdots} = \tfrac{1}{2}x_1 \left(T^{(2)}_{m-1}\{x_n\}\right.\\
&\qquad + \left. T^{(2)}_{m+1}\{x_n\}\right) + \ldots + \tfrac{1}{2}nx \left(T^{(2)}_{m-n}\{x_n\} + (-1)^{n+1} T^{(2)}_{m+n}\{x_n\}\right)\\
&\frac{\partial T^{(1)}_{m}\{x_n\}}{\partial x_p} = \tfrac{1}{2}\left(T^{(1)}_{m-p}\{x_n\} - T^{(1)}_{m+p}\{x_n\}\right),\\
&\frac{\partial T^{(2)}_{m}\{x_n\}}{\partial x_p} = \tfrac{1}{2}\left(T^{(2)}_{m-p}\{x_n\} + (-1)^{n} T^{(2)}_{m+p}\{x_n\}\right);
\end{aligned}\right\} \tag{5}$$

$$T^{(1)}_{m}\{x_n\} = \frac{-4}{\pi}\int_0^{\pi/2} \theta \cos\left(\tfrac{1}{2}m\pi - x_1\cos\theta + x_3\cos 3\theta - \ldots\right) \sin\left(m\theta - x_2\sin 2\theta + x_4\sin 4\theta - \ldots\right) d\theta\,; \tag{6}$$

$$T^{(2)}_{m}\{x_n\} = \frac{-4}{\pi}\int_0^{\pi/2} \theta\, e^{-x_2\cos 2\theta + x_4\cos 4\theta - \cdots} \cos\left(\tfrac{1}{2}m\pi - x_1\cos\theta + x_3\cos 3\theta - \ldots\right) \sin m\theta\, d\theta\,; \tag{7}$$

$$T^{(1)}_{m}\{x_n\} + 2\frac{\partial}{\partial m} J^{(1)}_{m}\{x_n\} + \pi E^{(1)}_{m}\{x_n\} = 0, \tag{8}$$

$$T^{(2)}_{m}\{x_n\} + 2\frac{\partial}{\partial m} J^{(2)}_{m}\{x_n\} + \pi E^{(2)}_{m}\{x_n\} = 0, \tag{9}$$

5. 1. Series for $T^{(1)}_{m}\{x_n\}$ and $T^{(2)}_{m}\{x_n\}$ in terms of $J^{(1)}_{m}\{x_n\}$'s and $J^{(2)}_{m}\{x_n\}$'s respectively.

From (1) and (2) of § 5 we shall show that

$$T^{(1)}_{m}\{x_n\} = \sum_{k=1}^{\infty} \frac{1}{k}\left(J^{(1)}_{m+2k}\{x_n\} - J^{(1)}_{m-2k}\{x_n\}\right), \tag{1}$$

$$T^{(2)}_{m}\{x_n\} = \sum_{k=1}^{\infty} \frac{1}{k}\left(J^{(2)}_{m+2k}\{x_n\} - J^{(2)}_{m-2k}\{x_n\}\right). \tag{2}$$

To prove (1) we substitute

$$-\sum_{k=1}^{\infty} \frac{\sin 2k\theta}{k} \text{ for } \theta - \tfrac{1}{2}\pi$$

in § 5 (1). Now, term by term integration shall be possible provided the series obtained by the above substitution converges uniformly in $0 \leqslant \theta \leqslant \pi$.

Since the above Fourier series does not converge uniformly near $\theta = 0$ and $\theta = \pi$, therefore, we need to justify the term by term integration. Let δ and ε be two arbitrarily small positive numbers. We can find an integer M_0 such that

$$\left| \left(\frac{\pi}{2} - \theta\right) - \sum_{k=1}^{M} \frac{\sin 2k\,\theta}{k} \right| < \varepsilon, \text{ where } \delta \leqslant \theta \leqslant \pi \text{ and } M > M_0.$$

Again, for all values of θ between 0 and π, we have

$$\left(\frac{\pi}{2} - \theta\right) - \sum_{k=1}^{M} \frac{\sin 2k\,\theta}{k} = \int_\theta^{\pi/2} (1+2\cos 2t + 2\cos 4t + \ldots + 2\cos 2Mt)\, dt$$

$$= \int_\theta^{\pi/2} \frac{\sin(2M+1)\,t}{t} \cdot \frac{t}{\sin t}\, dt$$

$$= \frac{\pi}{2} \int_\phi^{\pi/2} \frac{\sin(2M+1)\,t}{t}\, dt = \frac{\pi}{2} \int_{(2M+1)\phi}^{(M+\frac{1}{2})\pi} \frac{\sin x}{x}\, dx$$

for some value of ϕ between θ and $\pi/2$, by the second mean—value theorem, since $(t/\sin t)$ is a monotonic increasing function.

From the graph of ($\sin x/x$) it can also be seen that the last expression does not exceed $\frac{1}{2}\pi \int_0^\pi \frac{\sin x}{x}\, dx$ in absolute value ; let this value be $\pi A/2$, so that

$$\left| T_m^{(1)}\{x_n\} + \frac{2}{\pi} \sum_{k=1}^{M} \int_0^\pi \frac{\sin 2k\,\theta}{k} \sin(m\,\theta - x_1 \sin\theta - \ldots - x_n \sin n\theta)\, d\theta \right|$$

$$= \frac{2}{\pi} \left| \int_0^\pi \left(\left(\theta - \frac{\pi}{2}\right) + \sum_{k=1}^{M} \frac{\sin 2k\,\theta}{k} \right) \sin(m\theta - x_1 \sin\theta - \ldots - x_n \sin n\theta)\, d\theta \right|$$

$$\leqslant \frac{2}{\pi} \left\{ \int_0^\delta + \int_\delta^{\pi-\delta} + \int_{\pi-\delta}^{\pi} \right\} \left| \left(\theta - \frac{\pi}{2}\right) + \sum_{k=1}^{M} \frac{\sin 2k\,\theta}{k} \right| \left| \sin(m\theta - x_1 \sin\theta - \ldots - x_n \sin n\theta) \right| d\theta$$

$$< \frac{2}{\pi} (\pi A\delta + (\pi - 2\delta)\,\varepsilon)\, B < 2(A\delta + \varepsilon)\, B$$

where B is the upper bound of $|\sin(m\theta - x_1 \sin\theta - \ldots - x_n \sin n\theta)|$.

Since $2(A\delta + \varepsilon)\,B$ is arbitrarily small, the term by term integration is justified. Therefore.

$$T_m^{(1)}\{x_n\} = \frac{-2}{\pi}\sum_{k=1}^{\infty}\int_0^{\pi}\frac{\sin 2k\,\theta}{k}\sin(m\theta - x_1\sin\theta - \ldots - x_n\sin n\theta)\,d\theta$$

$$= \sum_{k=1}^{\infty}\frac{1}{k}\left(J_{m+2k}^{(1)}\{x_n\} - J_{m-2k}^{(1)}\{x_n\}\right),$$

which gives the required result (1), similarly (2) can also be proved.

In this way we shall extend the definition of various functions associated with J_μ (x) to the functions of n arguments $x_1, \ldots, x_n$ so that they may be associated with the function $J_\mu^{(1)}$ $\{x_n\}$ or $J_\mu^{(2)}$ $\{x_n\}$ as the case may be.

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EXISTENCE OF PERIODIC ORBITS OF THE THIRD KIND IN THE ELLIPTIC RESTRICTED PROBLEM OF THREE BODIES AND THE STABILITY OF THE GENERATING SOLUTION

By

RAM KISHORE CHOUDHRY

Department of Mathematics, L. S. College, Muzaffarpur, Bihar, India

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1. INTRODUCTION

This paper is an outcome of my study of the papers of Danby[7] and Szebehely[12] on the elliptic restricted problem of three bodies. Danby in his paper has studied the stability of the triangular points and Szebehely in his paper studied the stability of the collinear points in the elliptic restricted problem of three bodies. The present work started because of a curiosity to know the stability of the generating solution in the sense in which Danby and Szebehely took. While going through the papers of Batrakov[3], Pius[9], and Abalakin[1], I found that they were mainly restricted with the existence of periodic orbits of the second kind in circular restricted problem of three bodies and this restriction raised a keen interest in me to examine whether the word circular can be replaced by elliptic and the periodic orbits of the second kind by periodic orbits of the third kind in the above problem. This problem might be solved by Hill's method[2] which seems to be more useful for numerical work. Here I have followed Poincare's method which looked to me more attractive for a qualitative study.

Coming to the results obtained in the paper it is found in section 6 that the orthogonal intersection of the generating solution with the x-axis is a necessary condition for the existence of periodic solution which coincides with the condition obtained for symmetric periodic orbits[4] and whence it follows that a complete classification of the periodic orbits of the third kind can be studied as in[5].

2. FORMULATION OF THE PROBLEM.

Here the motion of a body with negligible mass is being considered in the gravitational field of two finite masses. Both the finite masses are assumed to be describing ellipses about their common centre of masses. Coordinate axes are chosen in such a way that the x–axis will always coincide with the line joining the primaries and the common centre of masses is taken for the origin.

With this selection of the axes it is proposed to prove the existence of periodic orbits of the third kind. This existence is shown in section 6. In section 7, the stability of the generating solution is being considered taking into account the disturbing mass upto $O(\sqrt{m'})$. Under this consideration it was possible to classify the periodic orbits as :

(*i*) Elliptic periodic orbits,

(*ii*) Hyperbolic periodic orbits, and

(*iii*) Parabolic periodic orbits.

Also conditions have been established for which the three types of periodic orbits may exist. The above classification is only the author's generalisation for 3–dimensional corordinate system of the similar orbits for two dimensional co-ordinate system.

3. EQUATIONS OF MOTION

Let m_0 and m' be the two finite masses at S and J. Let O be the centre of masses m_0 and m' and let $O\ \xi\ \eta\ \zeta$ be the coordinate axes such that $\xi\ \eta$ is the plane of motion of S and J and $O\zeta$ is a line perpendicular to $\xi\ \eta$ – plane. If (ξ_0, η_0, O) and (ξ_1, η_1, O) be the coordinates of S and J, then equations of motion of the negligible mass $P(\xi, \eta, \zeta)$ may be written as [2, page 254]:

$$\ddot{\xi} = \frac{\partial W}{\partial \xi}, \quad \ddot{\eta} = \frac{\partial W}{\partial \eta}, \quad \ddot{\zeta} = \frac{\partial W}{\partial \zeta} \qquad \ldots . \ (1)$$

where $W = k^2\ (m_0/r_0 + m'/r')$ and

$r_0^2 = (\xi - \xi_0)^2 + (\eta - \eta_0)^2 + \zeta^2$, $r'^2 = (\xi - \xi_1)^2 + (\eta - \eta_1)^2 + \zeta^2$, ξ_0, $\xi_1, \ldots .$ are given by

$$(m_0 + m')\ \xi_0 = - m' r_1 \cos v', \qquad (m_0 + m')\ \eta_0 = - m' r_1 \sin v',$$

$$(m_0 + m')\ \xi_1 = m' r_1 \cos v', \qquad (m_0 + m')\ \eta_1 = m' r_1 \sin v',$$

where $r_1 = SJ$ and v' is the angle made by SJ with the positive direction of $O\xi$. It will be assumed that

$$r_1 = a'(1 - e'^2)/(1 + e' \cos v')$$

where a' and e' are the semi-major axis and the eccentricity of the ellipse described by the relative motion of J *w. r.* to S.

Let us change our fixed axes $O\ \xi\ \eta\ \zeta$ to rotating axes $O\ x\ y\ z$ with the angular velocity $\dot{v}'$ where v' is the same angle as introduced before so that the x–axis will always coincide with the line joining SJ. If (x, y, z) be the coordinates of P relative to axes $O\ x\ y\ z$, then they will be given by the following transformation formulae [6, page 242]

$$\begin{aligned} \xi &= x \cos v' - y \sin v' \\ \eta &= x \sin v' + y \cos v' \\ \zeta &= z \end{aligned}$$

Equations of motion (1) reduce to

$$\begin{aligned} \ddot{x} - 2\dot{y}\,\dot{v}' - x\,\dot{v}'^2 - y\,\ddot{v}' &= \frac{\partial W}{\partial x} \\ \ddot{y} + 2\dot{x}\,\dot{v}' - y\,\dot{v}'^2 - x\,\ddot{v}' &= \frac{\partial W}{\partial y} \\ \ddot{z} &= \frac{\partial W}{\partial z} \end{aligned}$$

where $W = k^2\ (m_0/r_0 + m'/r')$ and

$$r_0^2 = (x - x_0)^2 + y^2 + z^2, \; r'^2 = (x - x_1)^2 + y^2 + z^2$$

$$x_0 = -\frac{m' r_1}{m_0 + m'}, \; x_1 = \frac{m_0 r_1}{m_0 + m'}$$

The expression for T, the kinetic energy per unit mass is given by
$T = \frac{1}{2} [\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + 2 \dot{v}' (x \dot{y} - y \dot{x}) + \dot{v}'^2 (x^2 + y^2)]$

Taking x, y, z for the generalised coordinates, Hamiltonian function can be written as:

$$H = \frac{1}{2} \left[\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - \dot{v}'^2 (x^2 + y^2) - 2k^2 \left(\frac{m_0}{r_0} + \frac{m'}{r'} \right) \right]$$

Let us return to fixed axes through m_0, where $(x_0, 0, 0)$ are the coordinates of m_0 w. r. to the origin at the centre of masses, so that

$$x = X \cos v' + Y \sin v' + x_0. \; y = Y \cos v' - X \sin v', \; z = Z.$$

Then $\quad \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 + 2 \dot{v}' (\dot{X}Y - \dot{Y}X) + \dot{v}'^2 (X^2 + Y^2)$
$\quad + 2 \dot{x}_0 (X \cos v' + Y \sin v') + \dot{x}_0^2.$

So $\quad H = \frac{1}{2} [\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 + 2 \dot{v}' (\dot{X}Y - \dot{Y}X) + 2 \dot{x}_0 (X \cos v' + Y \sin v') + \dot{x}_0^2$
$\quad - \dot{v}'^2 x_0 \{ 2(X \cos v' + Y \sin v') + x_0 \} - 2k^2 \left(\frac{m_0}{r_0} + \frac{m'}{r'} \right)]$

4. INTERMEDIATE ORBIT AND EQUATIONS OF MOTION IN DELAUNAY VARIABLES

Since 0 is the centre of masses m_0 and m' placed at $S(x_0, 0, 0)$ and $J(x_1, 0, 0)$, so

$$x_0 = -\frac{m' r_1}{m_0 + m'}, \; x_1 = \frac{m_0 r_1}{m_0 + m'}$$

Let the unit of masses be so chosen that $m_0 + m' = 1$ and let the two masses be m' and $1 - m'$. Also, let us assume the eccentricity e' of the elliptic orbits of the primaries be of $0\ (m')$, *i.e.*, let $e' = m'e_1$. Now by [11, Section 2·03]. $r_1^2 \dot{v}' = k.$ $\sqrt{a'(1 - e'^2)}$ and $n_1 = k/a'^{3/2}$, so neglecting the terms of $0\ (m'^2)$, $\dot{v}' = n_1 + 2n_1 m' e_1 \times \cos v'$. Again, $\dot{x}_0 = -m', \dot{r}_1 = m'^2 \frac{n_1 a' e_1 \sin v'}{\sqrt{1 - m'^2 e_1^2}} = O\ (m'^2)$. If ϑ be the angle subtended at S by the line joining J and P at any instant,

then $\quad \cos \vartheta = (X \cos v' + Y \sin v')/r.$

and so $\quad \dot{v}'^2 x_0 (X \cos v' + Y \sin v') = -k^2 m' \frac{r \cos \vartheta}{r_1^2} + 0\ (m'^2).$

Thus neglecting terms of $0\ (m'^2)$, H can be written as:

$$H = \frac{1}{2} \left[\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 + 2n_1 (\dot{X}Y - \dot{Y}X) - \frac{2k^2}{r} \right] - m' k^2 \left\{ \frac{1}{r'} - \frac{1}{r} - \frac{r \cos \vartheta}{r_1^2} \right\}$$
$$+ 2m' n_1 e_1 \cos v' (\dot{X}Y - \dot{Y}X) + 0\ (m'^2)$$

For the intermediate orbit, let us consider the Hamiltonian function H_0 which is given by

$$H_0 = \tfrac{1}{2}\left[\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 + 2n_1(\dot{X}Y - \dot{Y}X) - \frac{2k^2}{r}\right]$$

Let us change now our variables to spherical-polar coordinates r, θ and ϕ and $\dot{r}$, $\dot{\theta}$ and $\dot{\phi}$ by the following formulae:

$$x = r\cos\theta\cos\phi,\quad \dot{x} = \dot{r}\cos\theta\cos\phi - r\dot{\theta}\sin\theta\cos\phi - r\dot{\phi}\cos\theta\cos\phi$$

$$y = r\cos\theta\sin\phi,\quad \dot{y} = \dot{r}\cos\theta\sin\phi - r\dot{\theta}\sin\theta\sin\phi + r\dot{\phi}\cos\theta\cos\phi$$

$$z = r\sin\theta\quad,\ \dot{z} = \dot{r}\sin\theta\quad + r\dot{\theta}\cos\theta$$

Therefore

$$H_0 = \tfrac{1}{2}\left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\cos^2\theta\dot{\varphi}^2\right] - n_1 r^2\dot{\varphi}\cos^2\theta - \frac{k^2}{r}$$

We shall prefer to write our equations of motion in new set of variables r, θ and ϕ. For this, the other set of canonical variables will be introduced by the formulae:

$$p_1 = \frac{\partial T_0}{\partial \dot{r}} = \dot{r},\ p_2 = \frac{\partial T_0}{\partial \dot{\theta}} = r^2\dot{\theta},\ p_3 = \frac{\partial T_0}{\partial \dot{\varphi}} = r^2\dot{\varphi}\cos^2\theta$$

where $T_0 = \frac{1}{2}[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\cos^2\theta\,\dot{\varphi}^2]$

Thus $\quad H_0 = \frac{1}{2}[p_1^2 + p_2^2 r^{-2} + p_3^2 r^{-2}\sec^2\theta] - n_1 p_3 - k^2 r^{-1}$,

and the reduced form of the canonical equations are

$$\left.\begin{aligned} \frac{dr}{dt} &= \frac{\partial H_0}{\partial p_1},\ \frac{d\theta}{dt} = \frac{\partial H_0}{\partial p_2},\ \frac{d\varphi}{dt} = \frac{\partial H_0}{\partial p_3} \\ \frac{dp_1}{dt} &= -\frac{\partial H_0}{\partial r},\ \frac{dp_2}{dt} = -\frac{\partial H_0}{\partial \theta},\ \frac{dp_3}{dt} = -\frac{\partial H_0}{\partial \varphi} \end{aligned}\right\} \quad \ldots\ldots (2)$$

To find the solution of the above equations, consider the corresponding Hamilton-Jacobi equation,

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{r^2\cos^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2\right] - n_1\frac{\partial S}{\partial \varphi} - \frac{k^2}{r} = 0$$

The solution of this equation can be written as [11, pp. 143–148]:

$$S = -\alpha_1 t + \alpha_3\varphi + \int_{r_1}^{r} \sqrt{(2\alpha_1 + 2n_1\alpha_3)\,r^2 + 2k^2 r - \alpha_2^2}\,\frac{dr}{r}$$

$$+ \int_0^\theta \sqrt{\alpha_2^2 - \alpha_3^2\sec^2\theta}\,d\theta - \quad \ldots\ldots (3)$$

where r_1 and r_2 are the roots of the equation

$$(2\alpha_1 + 2n_1\alpha_3)\,r^2 + 2k^2 r - \alpha_2^2 = 0$$

and $0 < r_1 < r_2$. Hence the complete integral of (2) is furnished by (3) *i.e.*, by

$$\beta_1 = \frac{\partial S}{\partial \alpha_1}, \beta_2 = \frac{\partial S}{\partial \alpha_2}, \beta_3 = \frac{\partial S}{\partial \alpha_3}$$

$$p_1 = \frac{\partial S}{\partial r}, p_2 = \frac{\partial S}{\partial \theta}, p_3 = \frac{\partial S}{\partial \varphi}$$

As in [11, pp. 143-148], $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$, may be given the following values in terms of elliptic elements:

$$\alpha_1 = -\frac{k^2}{2a} - n_1 k \sqrt{a(1-e^2)} \cos \mathrm{I}, \quad \beta_1 = -\tau$$

$$\alpha_2 = k\sqrt{a(1-e^2)}, \qquad \beta_2 = \omega$$

$$\alpha_3 = k\sqrt{a(1-e^2)} \cos \mathrm{I}, \qquad \beta_3 = \Omega + n_1 (t-\tau)$$

For convenience sake let us introduce Delaunay variables instead of α_i, β_i, $(i = 1, 2, 3)$ for our further study. Delaunay variables are given as

$$L = k\sqrt{a}, \; G = k\sqrt{a(1-e^2)}, \; H = k\sqrt{a(1-e^2)} \cos \mathrm{I},$$

$$l = \frac{k^4}{L^3} (t-\tau), \; g = \omega \qquad , h = \Omega$$

Therefore $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2$, and β_3 may be expressed as follows:

$$\alpha_1 = -\frac{k^4}{2L^2} - n_1 H, \; \alpha_2 = G, \; \alpha_3 = H$$

$$\beta_1 = \frac{L^3 l}{k^4} - t, \; \beta_2 = g, \; \beta_3 = h + n_1 \frac{L^3}{k^4} l$$

Since

$$\left(\frac{L^3}{k^4} l - t\right)\left(\frac{k^4}{L^3} dL - n_1 dH\right) + gdG + \left(h + n_1 \frac{L^3}{k^4} l\right) dH - (ldL + gdG + hdH)$$
$$= td\left(\frac{k^4}{2L^2} + n_1 H\right)$$

so the canonical equations can be written in new variables in the form:

$$\frac{dL}{dt} = \frac{\partial R}{\partial l}, \; \frac{dG}{dt} = \frac{\partial R}{\partial g}, \; \frac{dH}{dt} = \frac{\partial R}{\partial h}$$

$$\frac{dl}{dt} = -\frac{\partial R}{\partial L}, \; \frac{dg}{dt} = -\frac{\partial R}{\partial G}, \; \frac{dh}{dt} = -\frac{\partial R}{\partial H}$$

where $R = \frac{k^4}{2L^2} + n_1 H + m' k^2 \left[\frac{1}{r'} - \frac{1}{r} - \frac{r \cos \vartheta}{r_1^2}\right] - 2m' n_1 e_1 H \cos v'$
$+ O(m'^2).$

5. EXPANSION OF THE DISTURBING FUNCTION AND THE ELIMINATION OF t.

Restricting ourselves with the terms of $O(m')$, we write from [2, pp. 490-494] that

$$\frac{1}{r'} - \frac{1}{r} - \frac{r \cos \vartheta}{r_1^2} = \frac{1}{k^2} \sum_{i_1, i_2, i_3} C^{(i_1, i_2, i_3)} (L, G, H) \cos (i_1 l + i_2 g + i_3 h)$$

It may be noted here that l', the mean anomaly of the primaries will not be involved in the arguments of the cosine functions for then the coefficients will have e' as a factor. But m' is already a factor and thus such terms will be $0\ (m'^2)$ and they will be neglected.

Similarly, $\cos v' = \cos l'$. Thus

$$R = \frac{k^4}{2L^2} + n_1 H + m' \sum_{i_1, i_2, i_3} C^{(i_1, i_2, i_3)} (L, G, H) \cos (i_1 l + i_2 g + i_3 h)$$
$$- 2m' e_1 n_1 H \cos l' + O\ (m'^2)$$

Now in order to avoid l' from the disturbing function, let us introduce the conjugate variable L'. Then as

$$\frac{dl'}{dt} = n_1 = - \frac{\partial R}{\partial L'}$$

and so let

$$R = \frac{k^4}{2L^2} + n_1 (H - L') + m' \Big[\sum_{i_1, i_2, i_3} C^{(i_1, i_2, i_3)} (L, G, H) \cos (i_1 l + i_2 g + i_3 h)$$
$$- 2n_1 e_1 H \cos l' \Big] + O\,(m'^2)$$

Let us now transform from the variables L, G, H, L', l, g, h, and l' to $\overline{L}, \overline{G}, \overline{H}, \ldots$ in such a way that our Hamiltonian gets rid of l'. For this we shall find out the determining function S given by $S = \overline{L}\, l + \overline{G} g + \overline{H} h + \overline{L}' l' + S_1 + S_2 + \ldots$

where $$L = \frac{\partial S}{\partial l}, \quad \overline{l} = \frac{\partial S}{\partial \overline{L}} \text{ etc.}$$

Let the transformed Hamiltonian be R^* and then

$$R \left(\overline{L} + \frac{\partial S_1}{\partial l} + \ldots, \overline{G} + \frac{\partial S_1}{\partial g} + \ldots, \overline{H} + \frac{\partial S_1}{\partial h} + \ldots, \overline{L'} + \frac{\partial S_1}{\partial l'} + \ldots,\right.$$
$$\left. l, g, h, l' \right) = R^* \left(\overline{L}, \overline{G}, \overline{H}, \overline{L'}, l + \frac{\partial S_1}{\partial \overline{L}} + \ldots, g + \right.$$
$$\left. + \frac{\partial S_1}{\partial \overline{G}} + \ldots, h + \frac{\partial S_1}{\partial \overline{H}} + \ldots, \right)$$

So equating the terms of equal orders on the two sides ;
Other order :

$$R_0 = R_0^* = \frac{k^4}{2\overline{L}^2} + n_1 (\overline{H} - \overline{L'})$$

1st order :

$$\frac{\partial S_1}{\partial l} \cdot \frac{\partial R_0^*}{\partial \overline{L}} + \frac{\partial S_1}{\partial h} \cdot \frac{\partial R_0^*}{\partial \overline{H}} + \frac{\partial S_1}{\partial l'} \cdot \frac{\partial R_0^*}{\partial \overline{L'}} - 2n_1 e_1 \overline{H} m' \cos l' = 0$$

The latter equation is satisfied by $S_1 = - 2e_1 \overline{H} m' \sin l'$.

Thus to the first order

$$R^* = \frac{k^4}{2\overline{L}^2} + n_1 (\overline{H} - \overline{L'}) + \sum_{i_1, i_2, i_3} C^{(i_1, i_2, i_3)} (\overline{L}, \overline{G}, \overline{H},) \cos (i_1 l + i_2 g + i_3 h)$$

where $\overline{L} = L, \overline{G} = G, \overline{H} = H, L' = \overline{L} - 2e_1 \overline{H} m' \cos l', \overline{l} = l, \overline{g} = g,$
$h = h - 2e_1 m' \sin l', \overline{l}' = l'$

Dropping bars (–), we can write the Hamiltonian as :

$$R = \frac{k^4}{2L^2} + n_1 (H - L') + \sum_{i_1, i_2, i_3} C^{(i_1, i_2, i_3)} (L, G, H) \cos (i_1 l + i_2 g + i_3 h)$$

where cosine function has been expanded in Taylor's series and the terms of $O(m'^2)$ have been neglected. It is now seen that l' does not appear in R and so for the further study L' may be regarded as a constant whereas $L, G, H, l, g,$ and h are variables.

Later on, it is seen that L' and l' have the same period as H and h, so for the existence of periodic solution, we shall have to show only that there is a periodic solution in $L, G, H, l, g,$ and h.

6. EXISTENCE OF PERIODIC ORBITS

For symmetry let us introduce new variables

$$x_1 = L,\ x_2 = G,\ x_3 = H$$
$$y_1 = l,\ y_2 = g,\ y_3 = h$$

Let us represent the disturbing function in the variables (x_i, y_i) $(i = 1, 2, 3)$ by F, where $F = F_0 + m'F_1 + m'^2 F_2 + \dots$

We shall now try to find the periodic solution of the third kind by Poincare's method[10] for the equations of motion

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i},\ \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i}\ (i = 1, 2, 3) \qquad \dots\dots (4)$$

For $m' = 0$, the equations (4) can be written as

$$\frac{dx_i}{dt} = 0,\ \frac{dy_i}{dt} = -\frac{\partial F_0}{\partial x_i}\ (i = 1, 2, 3)$$

The generating solution $x_i^{(0)} = a_i, y_i^{(0)} = n_i^{(0)} t + \omega_i$ are well-known solution of the problem of two bodies. Here a_i and ω_i are arbitrary constants and

$$n_1^{(0)} = -\frac{\partial F_0}{\partial x_i}\Big|_{x_i = a_i},\ n_2^{(0)} = 0 = -\frac{\partial F_0}{\partial x_2}\Big|_{x_i = a_i},\ n_3^{(0)} = -n_1 = -\frac{\partial F_0}{\partial x_3}\Big|_{x_i = a_i}$$

This solution will be periodic with the period T_0, if

$$y_i(T_0) - y_i(0) = n_i^{(0)} T_0 = 2k_i\pi\ (i = 1, 2, 3)$$
$$x_i(T_0) - x_i(0) = 0$$

Here k_i $(i = 1, 2, 3)$ are integers, so that $n_i^{(0)}$ are commensurable.

Let the general solution in the neighbourhood of the generating solution be periodic with the period $T_0(1+\alpha)$ where α is a negligible quantity of the order m'. Introduce new independent variable ζ by the formula :

$$\zeta = \frac{t}{1+\alpha}$$

for which the period of the general solution will be T_0 which coincides with that of the generating solution. Therefore, the equations of motion (4) can be written as

$$\frac{dx_i}{d\zeta} = (1+\alpha)\frac{\partial F}{\partial y_i}, \quad \frac{dy_i}{d\zeta} = -(1+\alpha)\frac{\partial F}{\partial x_i} \; (i = 1, 2, 3) \qquad \ldots. \quad (5)$$

and the general solution in the neighbourhood of the generating solution may be given as

$$x_i = a_i + \beta_i + \xi_i(\zeta), \; y_i = n_i^{(0)} \zeta + \omega_i + \gamma_i + \eta_i(\zeta) \; (i = 1, 2, 3).$$

The equations (5) can be written in terms of new variables ξ_i and η_i as

$$\frac{d\xi_i}{d\zeta} = \frac{\partial K}{\partial \eta_i}, \quad \frac{d\eta_i}{d\zeta} = -\frac{\partial K}{\partial \xi_i} \; (i = 1, 2, 3) \qquad \ldots. \quad (6)$$

where

$$\begin{aligned} K(\zeta \mid \xi_i \mid \eta_i) = {} & \xi_1 n_1^{(0)} + \xi_3 n_3^{(0)} + \alpha F(\zeta \mid a_0 + \beta_i \mid n_i^{(0)}\zeta + \omega_i + \gamma_i) \\ & + (1+\alpha)\left[\sum_{i=1}^{3} \frac{\partial F}{\partial a_i}\xi_i + \sum_{i=1}^{3} \frac{\partial F}{\partial \omega_i}\eta_i\right] \\ & + \tfrac{1}{2}(1+\alpha)\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\xi_i\xi_j \frac{\partial^2 F}{\partial a_i\, \partial a_j} + \xi_i \eta_j \frac{\partial^2 F}{\partial a_i\, \partial \omega_j} + \eta_i \eta_j \frac{\partial^2 F}{\partial \omega_i\, \partial \omega_j}\right) \\ & + O(\xi_i^3) \qquad \ldots. \quad (7) \end{aligned}$$

Now in order that the periodic solution of the third kind may exist, the necessary and sufficient conditions are written as [10, pp. 134] :

$$x_i(T_0) - x_i(0) = \xi_i(T_0) = 0 \qquad \ldots. \quad (8)$$

$$y_i(T_0) - y_i(o) - 2k_i\pi = \eta_i(T_0) = 0 \qquad \ldots. \quad (9)$$

Restricting our solution only upto the 1st order infinitesimals, the equations (6) may be written as

$$\frac{d\xi_i}{d\zeta} = (1+\alpha)\frac{\partial F}{\partial \omega_i}, \quad \frac{d\eta_i}{d\zeta} = -\eta_i^{(0)} - (1+\alpha)\frac{\partial F}{\partial a_i} \; (i = 1, 2, 3) \qquad \ldots. \quad (10)$$

Now expanding $F(\zeta/a_0 + \beta_i/\eta_i^{(0)}\zeta + \omega_i + \gamma_i)$ in ascending powers of $\beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3, m'$ we find that the equations (8) and (9) may be written as :

$$\frac{\xi_k(T_0, \beta_i, \gamma_i, m')}{m' T_0} = \frac{\partial [F_1]}{\partial \omega_k} + \sum_{j=1}^{3} \frac{\partial^2 [F_1]}{\partial \omega_k \partial a_j} \beta_j + \sum_{j=1}^{3} \frac{\partial^2 [F_1]}{\partial \omega_k \partial \omega_j} \gamma_j + \ldots = 0$$

$$(k \neq 1, 2, 3)$$

$$\eta_1(T_0, \beta i, \gamma i, m') = -T_0 \alpha \frac{\partial F_0}{\partial a_1} - m' \left[T_0 \frac{\partial [F_1]}{\partial a_1} - \int_0^{T_0} \frac{\partial^2 F_0}{\partial a_1^2} \int_0^t \frac{\partial F_1}{\partial \omega_1} \zeta d \zeta dt \right]$$

$$- \beta_1 T_0 \left[\frac{\partial^2 F_0}{\partial a_1^2} + m' \frac{\partial^2 [F_1]}{\partial a_1^2} - \frac{1}{T_0} \frac{\partial^2 F_0}{\partial a_1^2} \int_0^{T_0} \frac{\partial^2 F_1}{\partial a_1 \partial \omega_1} t\, dt \right]$$

$$- m' T_0 \left[\beta_2 \frac{\partial^2 [F_1]}{\partial a_1 \partial a_2} + \beta_3 \frac{\partial^2 [F_1]}{\partial a_1 \partial a_3} \right.$$

$$\left. + \gamma_1 \frac{\partial^2 [F_1]}{\partial a_1 \partial \omega_1} + \gamma_2 \frac{\partial^2 [F_1]}{\partial a_1 \partial \omega_2} + \gamma_3 \frac{\partial^2 [F_1]}{\partial a_1 \partial \omega_3} \right] + \ldots = 0$$

$$\eta_2(T_0, \beta i, \gamma i, m') = -m' T_0 \left[\frac{\partial [F_1]}{\partial a_2} + \beta_1 \left(\frac{\partial^2 [F_1]}{\partial a_2 \partial a_1} - \frac{1}{T_0} \frac{\partial^2 F_0}{\partial a_1^2} \int_0^{T_0} \frac{\partial^2 F_1}{\partial a_2 \partial \omega_1} t\, dt \right) \right. \qquad (11)$$

$$+ \beta_2 \frac{\partial^2 [F_1]}{\partial a_2^2} + \beta_3 \frac{\partial^2 [F_1]}{\partial a_2 \partial a_3} +$$

$$\left. + \gamma_1 \frac{\partial^2 [F_1]}{\partial a_2 \partial \omega_1} + \gamma_2 \frac{[\partial^2 F_1]}{\partial a_2 \partial \omega_2} + \gamma_3 \frac{\partial^2 [F_1]}{\partial a_2 \partial \omega_3} + \ldots \right] = 0$$

$$\eta_3(T_0, \beta i, \gamma i, m') = -T_0 \alpha \frac{\partial F_0}{\partial a_3} - m' T_0 \frac{\partial [F_1]}{\partial a_3} - m' T_0 \left[\beta_1 \left(\frac{\partial^2 [F_1]}{\partial a_3 \partial a_1} \right. \right.$$

$$\left. - \frac{1}{T_0} \frac{\partial^2 F_0}{\partial a_1^2} \int_0^{T_0} \frac{\partial^2 F_1}{\partial a_3 \partial \omega_1} t\, dt \right)$$

$$+ \beta_2 \frac{\partial^2 [F_1]}{\partial a_3 \partial a_2} + \beta_3 \frac{\partial^2 [F_1]}{\partial a_3^2}$$

$$\left. + \gamma_1 \frac{\partial^2 [F_1]}{\partial a_3 \partial \omega_1} + \gamma_2 \frac{\partial^2 [F_1]}{\partial a_3 \partial \omega_2} + \gamma_3 \frac{\partial^2 [F_1]}{\partial a_3 \partial \omega_3} \right] + \ldots = 0$$

where $[F_1] = \frac{1}{T_0} \int_0^{T_0} F_1 (\zeta / ai / \eta_i^{(0)} \zeta + \omega_i)\, d\zeta \;\; (i = 1, 2, 3).$

Since ζ is not involved explicitely in the characteristic function F, so $F = C$ is an integral and $\frac{\partial F}{\partial x_1} \neq 0$, the first of the above six equations may be considered as a consequence of the remaining five. Thus for seven unknowns β_i, γ_i and α we shall have only five equations. Hence two unknowns (say γ_1 and α) may be chosen arbitrarily. Let $\gamma_1 = 0$ and $\alpha = \alpha(m') \neq 0$ and again as the choice of the origin of time is arbitrary, so we may take $\omega_1 = 0$. Then the equations (11) will give β_1, β_2, β_3, γ_2 and γ_3 as holomorphic functions of m', reducing to zero with m', if the following conditions [6, pp. 178–184] are satisfied :

$$\frac{\partial[F_1]}{\partial\omega_i} = 0 \ (i = 1, 2, 3) \qquad \dots . (12)$$

$$\frac{\partial[F_1]}{\partial a_i} = 0 \ (i = 1, 2, 3) \qquad \dots . (13)$$

$$\left.\frac{\partial\,(\xi_2, \xi_3, \eta_1, \eta_2, \eta_3)}{\partial(\gamma_2, \gamma_3, \beta_1, \beta_2, \beta_3)}\right| \begin{array}{l} \neq 0 \\ m' = \beta i = \gamma_i = 0 \ (i = 1, 2, 3) \end{array} \qquad \dots . (14)$$

The Jacobian on left hand side of (14) can be written as

$$\begin{vmatrix} 0 & 0 & 0 & \frac{\partial^2[F_1]}{\partial\omega_2^2} & \frac{\partial^2[F_1]}{\partial\omega_2\,\partial\omega_3} \\ 0 & 0 & 0 & \frac{\partial^2[F_1]}{\partial\omega_3\,\partial\omega_2} & \frac{\partial^2[F_1]}{\partial\omega_3^2} \\ \frac{\partial^2 F_0}{\partial a_1^2} & 0 & 0 & 0 & 0 \\ - & \frac{\partial^2[F_1]}{\partial a_2^2} & \frac{\partial^2[F_1]}{\partial a_2\,\partial a_3} & 0 & 0 \\ - & \frac{\partial^2[F_1]}{\partial a_3\,\partial a_2} & \frac{\partial^2[F_1]}{\partial a_3^2} & 0 & 0 \end{vmatrix} \neq 0$$

and so the condition (14) reduces to

$$\frac{\partial^2 F_0}{\partial a_1^2} \begin{vmatrix} \frac{\partial^2[F_1]}{d\omega_2^2} & \frac{\partial^2[F_1]}{\partial\omega_2\,\partial\omega_3} \\ \frac{\partial^2[F_1]}{\partial\omega_2\,d\omega_3} & \frac{\partial^2[F_1]}{\partial\omega_3^2} \end{vmatrix} \times \begin{vmatrix} \frac{\partial^2[F_1]}{\partial a_2^2} & \frac{\partial^2[F_1]}{\partial a_2\,\partial a_3} \\ \frac{\partial^2[F_1]}{\partial a_3\,\partial a_2} & \frac{\partial^2[F_1]}{\partial a_3^2} \end{vmatrix} \neq 0 \qquad \dots . (14)$$

[It may be noted here that $\frac{\partial^2[F_1]}{\partial a_j\,\partial\omega_i} = 0$ for $\frac{\partial[F_1]}{\partial\omega_i} = 0$ ($i = 1, 2, 3$; $j = 1, 2, 3$)].

The conditions (13) reduce to

$$\left.\begin{array}{l} \frac{1-e^2}{e}\frac{\partial[F_1]}{\partial e} - \frac{1-2\sigma^2}{4\sigma}\frac{\partial[F_1]}{\partial\sigma} = 0 \\ -\frac{1}{4\sigma\sqrt{a(1-e^2)}}\frac{\partial[F_1]}{\partial\sigma} = 0 \end{array}\right\} \qquad \dots . (15)$$

where $\sigma = \sin I/2$, gives e and σ for which the periodic solution may exist. Let e_0 and σ_0 be the corresponding values.

Since $F_0 = \frac{k^4}{2a_1^2} + n_1 a_3$ and so $\frac{\partial^2 F_0}{\partial a_1^2} \neq 0$ .... (16)

Now the existence of the periodic solution will follow if the two determinants occuring in (14) are not zero.

Since

$$F_1 = \sum_{i_1, i_2, i_3} C_{i_1, i_2, i_3}(a_1, a_2, a_3) \cos [i_1 (n_1^{(0)} \zeta + \omega_1) + i_2 \omega_2 + i_3 (n_3^{(0)} \zeta + \omega_3)]$$

$$\therefore [F_1] = \sum_{\substack{i_1, i_2, i_3 \\ i_1 n_1^{(0)} + i_3 n_3^{(0)} = 0}} C_{i_1, i_2, i_3}(a_1, a_2, a_3) \cos [i_1 \omega_1 + i_2 \omega_2 + i_3 \omega_3]$$

The conditions (12) show that $\frac{\partial[F_1]}{\partial\omega_2} = 0 = \sin(i_2\omega_2 + i_3\omega_3) = \frac{\partial[F_1]}{\partial\omega_3}$. So either $\omega_2 = \omega_3 = 0$; or $\omega_2 = 0$, $\omega_3 = \pi$; or $\omega_2 = \omega_3 = \pi$; or $\omega_2 = \pi$, $\omega_3 = 0$; which show that the conditions of symmetric conjunctions or oppositions are the necessary conditions for the existence of the periodic solutions which coincide with the conditions obtained in [4, Sec. 3]. For definiteness, let $\omega_2 = \omega_3 = 0$ in which case, we find that

$$\left.\frac{\partial^2[F_1]}{\partial\omega_2^2}\right|_{\omega_2 = \omega_3 = 0} \neq 0, \qquad \left.\frac{\partial^2[F_1]}{\partial\omega\ \partial\omega j}\right|_{\omega_2 = \omega_3 = 0} \neq 0 \qquad (i = 1, 2, 3 ; j = 1, 2, 3 ; i \neq j)$$

Now expanding $[F_1]$ in the neighbourhood of $\omega_2 = 0 = \omega_3$, we find that

$$[F_1](\omega_2, \omega_3) - [F_1](0, 0) = \tfrac{1}{2} \frac{\partial^2[F_1]}{\partial\omega_2^2} \omega_2^2 + \frac{\partial^2[F_1]}{\partial\omega_2\, \partial\omega_3} \omega_2 \omega_3 + \tfrac{1}{2} \frac{\partial^2[F_1]}{\partial\omega_3^2} \omega_3^2$$

Since $[F_1](0, 0)$ is a stationary value, so the L. H. S. must be of constant sign and so

$$\frac{\partial^2[F_1]}{\partial\omega_2^2} \frac{\partial^2[F_1]}{\partial\omega_3^2} - \left(\frac{\partial^2[F_1]}{\partial\omega_2\, \partial\omega_3}\right)^2 > 0 \qquad \ldots. \quad (17)$$

i. e. the 1st determinant on the L. H. S. of (14) is not equal to zero. For the other determinant, let

$$[F_1] = \{\tfrac{1}{2} A_0 + \tfrac{1}{8}\left(1 - \frac{a_2^2}{a_1^2} + e'^2\right)[2\alpha D_\alpha + \alpha^2 D_\alpha^2]\, A_0 - \tfrac{1}{8}\left(1 - \frac{a_3}{a_2}\right)(B_{-1} + B_{+1})\} + \ldots$$

where the expression is taken from [2, pp. 490–494]. Since the generating solution is an ellipse, so $e \neq 1$, *i. e.* $a_2 \neq 0$ and thus $[F_1]$ is a continuous function of a_2 and a_3 and so $[F_1]$ will attain its extremum in $(0 < a_2 \leqslant a_1)$, $(0 \leqslant a_3 \leqslant a_1)$ and therefore as in (17) it is clear that

$$\frac{\partial^2[F_1]}{\partial a_2^2} \frac{\partial^2[F_1]}{\partial a_3^2} - \frac{\partial^2[F_1]}{\partial a_2\, \partial a_3} > 0 \qquad \ldots.. \quad (18)$$

for $a_2 = \bar{a}_2$, $a_3 = \bar{a}_3$ ($\bar{a}_2$, $\bar{a}_3$ being the values corresponding to $e = e_0$ and $\sigma = \sigma_0$).

Thus combining (16), (17) and (18), we find that the determinant (14) is not equal to zero. Hence for a particular commensurability $\frac{n_1^{(0)}}{n_3^{(0)}} = -\frac{p}{q}$, there will exist a periodic solution of the third kind if e, e' and σ are given by the equations (15).

Since in every neighbourhood of $m' = 0$, there is an enumerable of m' whose limit is zero and so we may conclude that in every neighbourhood of $m' = 0$, there is an enumerable no. of periodic orbits of the third kind.

7. Stability of the generating solution

Definition [7]—Let C be an orbit. We say that C is *orbitally stable*, if given $\varepsilon > 0$, there is $\eta > 0$ such that, if R is a representative point of another trajectory which is within a distance η of C at time τ, then R remains within a distance ε of C for $t > \tau$. If no such η exists, C is *orbitally unstable.*

We shall be making use of the following theorems [6] which are given here without their proofs.

Theorem 1.—If the differential equations of the perturbed motion are such that it is possible to find a definite function V, whose differential $\dot{V}$ by virtue of these equations be of sign opposite to that of V or is identically zero, then the unperturbed motion is stable.

Theorem 2.—If the characteristic equation $D(s) = 0$, not possessing roots with positive real parts, has roots whose real parts are equal to zero, then the functions X_s can always be so chosen that the unperturbed motion is stable or unstable depending upon our choice.

[If in the expression (7) of section 4, terms of $0\ (\zeta_i^3)$ be also taken into consideration, then the corresponding terms in equations (10) will be denoted by X_s].

Now keeping the above theorem in our view let us find the characteristic exponents in the present problem. For this we shall reduce our problem from sixth order to a problem of the 4th order.

Characteristic Exponents

Canonical equations of the elliptic restricted problem of three bodies can be written in the form :

$$\left.\begin{aligned} \frac{dL}{dt} &= \frac{\partial F}{\partial l} & \qquad \frac{dl}{dt} &= -\frac{\partial F}{\partial L} \\ \frac{dG}{dt} &= \frac{\partial F}{\partial g} & \qquad \frac{dg}{dt} &= -\frac{\partial F}{\partial G} \\ \frac{dH}{dt} &= \frac{\partial F}{\partial h} & \qquad \frac{dh}{dt} &= -\frac{\partial F}{\partial H} \end{aligned}\right\} \quad \ldots . \quad (19)$$

where $F = F(L, G, H, l, g, h) = F_0(L, H) + m' F_1(L, G, H, l, g, h) + m'^2 F_2 + \cdots$

$$F_0(L, H) = k^4/2L^2 + n_1 H,$$

$$F_s = \sum_{i_1, i_2, i_3} C_s^{(i_1, i_2, i_3)}(L, G, H) \cos(i_1 l + i_2 g + i_3 h) \quad (s = 1, 2, \ldots .)$$

$L, G, H, \ldots$ have the same meaning as in section 4.

Since F does not involve t explicitly, so Jacobi's integral can be written as

$$F(L, G, H, l, g, h) = C \quad \ldots . \quad (20)$$

where C is the Jacobi's constant. As R. H. S. of (19) are independent of t, so instead of the independent variable t, we can introduce l which can be done as follows

$$\left.\begin{aligned}
\frac{dL}{dl} &= -\frac{\frac{\partial F}{\partial l}}{\frac{\partial F}{\partial L}} & t &= -\int \frac{dl}{\frac{\partial F}{\partial L}} \\
\frac{dG}{dl} &= -\frac{\frac{\partial F}{\partial g}}{\frac{\partial F}{\partial L}} & \frac{dg}{dl} &= \frac{\frac{\partial F}{\partial G}}{\frac{\partial F}{\partial L}} \\
\frac{dH}{dl} &= -\frac{\frac{\partial F}{\partial h}}{\frac{\partial F}{\partial L}} & \frac{dh}{dl} &= \frac{\frac{\partial F}{\partial H}}{\frac{\partial F}{\partial L}}
\end{aligned}\right\} \qquad \ldots. \quad (21)$$

Let us solve the equation (20) for L and thus $L = \Phi\,(C, G, H, l, g, h)$ where $\Phi\,(C, G, H, l, g, h) = \Phi_0\,(C, H) + m'\,\Phi_1\,(C, G, H, l, g, h) + \ldots.$

$$\Phi_0\,(C, H) = k^2/\sqrt{2C - 2n_1 H}$$

Φ_1 = a periodic function in l, g and h with the period 2π. Differentiating $F\,(\Phi, G, H, l, g, h) = C$ w. r. to l, g, h, G and H we shall get

$$\frac{\partial F}{\partial L}\cdot\frac{\partial \Phi}{\partial l} + \frac{\partial F}{\partial l} = 0, \quad \frac{\partial F}{\partial L}\cdot\frac{\partial \Phi}{\partial g} + \frac{\partial F}{\partial g} = 0, \quad \frac{\partial F}{\partial L}\cdot\frac{\partial \Phi}{\partial h} + \frac{\partial F}{\partial h} = 0$$

$$\frac{\partial F}{\partial L}\cdot\frac{\partial \Phi}{\partial G} + \frac{\partial F}{\partial G} = 0, \quad \frac{\partial F}{\partial L}\cdot\frac{\partial \Phi}{\partial H} + \frac{\partial F}{\partial H} = 0$$

Therefore, by means of (21) we shall get

$$\frac{dL}{dl} = \frac{\partial \Phi}{\partial l}, \quad \frac{dG}{dl} = \frac{\partial \Phi}{\partial g}, \quad \frac{dH}{dl} = \frac{\partial \Phi}{\partial h}$$

$$\frac{dg}{dl} = -\frac{\partial \Phi}{\partial G}, \quad \frac{dh}{dl} = -\frac{\partial \Phi}{\partial H}$$

Now the first equation coincides with $L = \Phi$ and so we find that the equations (19) are reduced to a system of four equations of the 1st order. Thus

$$\frac{dG}{dl} = m'\frac{\partial \Phi_1}{\partial g}, \quad \frac{dg}{dl} = -m'\frac{\partial \Phi_1}{\partial G}$$

$$\frac{dH}{dl} = m'\frac{\partial \Phi_1}{\partial h}, \quad \frac{dh}{dl} = -\frac{d\Phi_0}{dH} - m'\frac{\partial \Phi_1}{\partial H}$$

(Terms of $0(m'^2)$ are neglected).

For $m' = 0$, this system has got the solutions

$$G^{(0)} = G_0, \qquad H^{(0)} = H_0$$

$$g^{(0)} = g_0, \qquad h^{(0)} = -\frac{n_1}{n^{(0)}}\,l + h_0 \text{ where } n^{(0)} = \frac{(2C - 2n_1 H^{(0)})^{3/2}}{k^2}$$

Let T_0 be the period of the generating solution and thus

$$l(T_0) - l(0) = 2k_1\pi$$

and so

$$h^{(0)}(T_0) - h^{(0)}(0) = -\frac{n_1}{n^{(0)}} 2k_1 \pi = -2k_2 \pi$$

$\therefore n^{(0)}(H_0) = n_1 \frac{k_1}{k_2}$, where (k_1, k_2) are characteristic numbers of this periodic solution. According to the method of small parameters of Poincare, the general solution $G(l)$, $H(l)$, $g(l)$ and $h(l)$ can be given in terms of l and differences of the initial values for the unknowns for $m' = 0$ and for $m' \neq 0$. Let the general solution for $m' \neq 0$ in the neighbourhood of the generating solution be given as

$$G(l) = G_0 + \beta_1 + x_1, \quad g(l) = g_0 + \gamma_1 + y_1$$

$$H(l) = H_0 + \beta_2 + x_2, \quad h(l) = -\frac{n_1}{n^{(0)}} l + h_0 + \gamma_2 + y_2$$

and

$$G(0) = G_0 + \beta_1, \qquad g(0) = g_0 + \gamma_1$$
$$H(0) = H_0 + \beta_2, \qquad h(0) = h_0 + \gamma_2.$$

Let $x_1(\beta, \gamma, m') = G(T) - G(0)$, $y_1(\beta, \gamma, m') = g(T) - g(0)$,

$x_2(\beta, \gamma, m') = H(T) - H(0)$, $y_2(\beta, \gamma, m') = h(T) - h(0) + 2k_2\pi$.

Since we are concerned with only the characteristic exponents, so there is no loss of generality if we assume that the general solution has got the same period as for the generating solution. Taking into view the period for $h(l)$, let $T = 2k_2 \pi$.

As in [8], to the first order of β, γ, m'

$$x_1 = 2k_2 \pi m' \left(\frac{\partial[\Phi_1]}{\partial g_0} + \dots\dots\right), \quad x_2 = 2k_2 \pi m' \left(\frac{\partial[\Phi_1]}{\partial h_0} + \dots\dots\right)$$

$$y_1 = -2k_2\pi m' \left(\frac{\partial[\Phi_1]}{\partial G_0} + \dots\dots\right)$$

$$y_2 = -2k_2 \pi \frac{d^2\Phi_0(H_0)}{dH_0^2} \beta_2 - m' \left[\int_0^{2k_2\pi} \frac{\partial\Phi_1(G_0, H_0, g^{(0)}, h^{(0)})}{\partial H_0} dl + \frac{d^2\Phi_0(H_0)}{dH_0^2} \int_0^{2k\pi} \left(\int_0^t \frac{\partial\Phi_1}{\partial h_0} d\tau \right) dt \right] + \dots$$

The condition of periodicity $x_1 = x_2 = y_1 = y_2 = 0$, gives the equations

$$\frac{\partial[\Phi_1]}{\partial g_0} = \frac{[\partial\Phi_1]}{\partial h_0} = \frac{\partial[\Phi_1]}{\partial G_0} = 0$$

which determine g_0, h_0 and G_0. According to Poincare[10], the characteristic exponents of the corresponding periodic solution, holomorphic in m', are given by the equation

$$\begin{vmatrix} \frac{\partial x_1}{\partial \beta_1} - S & \frac{\partial x_1}{\partial \beta_2} & \frac{\partial x_1}{\partial \gamma_1} & \frac{\partial x_1}{\partial \gamma_2} \\ \frac{\partial x_2}{\partial \beta_1} & \frac{\partial x_2}{\partial \beta_2} - S & \frac{\partial x_2}{\partial \gamma_1} & \frac{\partial x_2}{\partial \gamma_2} \\ \frac{\partial y_1}{\partial \beta_1} & \frac{\partial y_1}{\partial \beta_2} & \frac{\partial y_1}{\partial \gamma_1} - S & \frac{\partial y_1}{\partial \gamma_2} \\ \frac{\partial y_2}{\partial \beta_1} & \frac{\partial y_2}{\partial \beta_2} & \frac{\partial y_2}{\partial \gamma_1} & \frac{\partial y_2}{\partial \gamma_1} - S \end{vmatrix} = 0 \qquad \dots\dots (22)$$

where $S = e^{2k_2 \alpha \pi} - 1$.

Let us assume that for $\beta_i = \gamma_i = 0$ $(i = 1, 2)$,

$$\frac{\partial x_1}{\partial \beta_1} = m' A_{11}(m'), \frac{\partial x_1}{\partial \beta_2} = m' A_{21}(m'), \frac{\partial x_1}{\partial \gamma_1} = m' A_{31}(m'), \frac{\partial x_1}{\partial \gamma_2} = m' A_{41}(m')$$

$$\frac{\partial x_2}{\partial \beta_1} = m' A_{12}(m'), \frac{\partial x_2}{\partial \beta_2} = m' A_{22}(m'), \frac{\partial x_2}{\partial \gamma_1} = m' A_{32}(m'), \frac{\partial x_2}{\partial \gamma_2} = m' A_{42}(m')$$

$$\frac{\partial y_1}{\partial \beta_1} = m' A_{13}(m') \frac{\partial y_1}{\partial \beta_2} = m' A_{23}(m'), \frac{\partial y_1}{\partial \gamma_1} = m' A_{33}(m'), \frac{\partial y_1}{\partial \gamma_2} = m' A_{43}(m')$$

$$\frac{\partial y_2}{\partial \beta_1} = m' A_{14}(m'), \frac{\partial y_2}{\partial \beta_2} = m' A_{21}(m') - 2k_2 \pi \frac{d^2\Phi_0}{dH_0^2}; \frac{dy_2}{\partial \gamma_1} = m' A_{34}(m'), \frac{\partial y_2}{\partial \gamma_2} = m' A_{44}(m')$$

Putting these values in the characteristic equation (22) and writing A_{11}, A_{21}, A_{31}, ... *for* A_{11} (m') ...., (22) can be written as

$$\begin{vmatrix} m'A_{11} - S & m'A_{21} & m'A_{31} & m'A_{41} \\ m'A_{12} & m'A_{22} - S & m'A_{32} & m'A_{42} \\ m'A_{13} & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ m'A_{14} & \cdot\cdot & \cdot\cdot & \cdot\cdot \end{vmatrix} = 0$$

Neglecting the terms of $0(m'^2)$, the above equation reduces to

$$(m'A_{11} - S)\ (m'A_{33} - S) \left[(m'A_{22} - S)\ (m'A_{41} - S) + m'A_{42} \left(- 2k_2 \pi \frac{d^2\Phi_0}{dH_0^2} \right) \right] = 0$$

Two roots are $m'A_{11}$ and $m'A_{33}$ and the other two roots are given by

$$S = \pm\ 2k_2 \pi \sqrt{- \frac{d^2\Phi_0}{dH_0^2} \cdot \frac{\partial^2[\Phi_1]}{\partial h_0^2} m'} + 0(m')$$

Restricting our consideration with the terms of $0(\sqrt{m'})$, we find that two characteristic numbers are zero and the other two are given by

$$S = \pm\ 2k_2 \pi \frac{\sqrt{3}\, n_1 k}{(2c - 2n_1 H_0)^{5/4}} \sqrt{- \frac{\partial^2[\Phi_1]}{\partial h_0^2} m'} + 0(m')$$

As S is very small, we may replace S by $2k_2 \pi \alpha$, then

$$\alpha = \pm \frac{\sqrt{3}\, n_1 k}{(2C - 2n_1 H_0)^{5/4}} \sqrt{- \frac{\partial^2[\Phi_1]}{\partial h_0^2} m'} + 0(m')$$

Thus the characteristic exponents are 0, 0, $+\alpha$ and $-\alpha$. Now

$$[\Phi_1] = \sum_{\substack{i_1, i_2, i_3 \\ i_1 n^{(0)} = i_3 n_1}} C^{(i_1, i_2, i_3)}(C, G_0, H_0) \cos (i_2 g_0 + i_3 h_0)$$

It is clear that $[\Phi_1]$ is a continuous function of h_0 and so it must attain its maximum and minimum in the closed interval $[0, 2\pi]$, *i.e.*, in the open interval $(-\varepsilon, 2\pi + \varepsilon)$ in h_0 and g_0 for an arbitrary $\varepsilon > 0$. Thus $[\Phi_1]$ has got in the interval $[0, 2\pi]$ at least one maximum and one minimum. The corresponding values h_0 will satisfy the periodicity conditions :

$$\frac{\partial[\Phi_1]}{\partial h_0} = 0 \text{ and } \frac{\partial^2[\Phi_1]}{\partial h_0^2} \neq 0$$

It will have maximum value when $\frac{\partial^2[\Phi_1]}{\partial h_0^2} < 0$ and minimum when $\frac{\partial^2[\Phi_1]}{\partial h_0^2} > 0$. The condition $\frac{\partial^2[\Phi_1]}{\partial h_0^2} = 0$ is not possible. Corresponding to the maximum value the characteristic exponents α will be real and corresponding to the minimum value α will be purely imaginary.

If α's are real, then the unperturbed orbit is unstable and if α's are imaginary, then by theorem 2, the stability or instability will depend upon the choice of higher order terms. We shall not discuss this later case.

Let us call the periodic orbits with purely imaginary characteristic exponents, elliptic orbits in the generalised sense and the orbits with non-zero real characteristic exponents, the hyperbolic periodic orbits and with zero characteristic exponents the parabolic periodic orbits.

For sufficiently small m', there exists elliptic periodic orbits as well as hyperbolic periodic orbits in a sufficiently small neighbourhood of the generating solution of the elliptic restricted problem of three bodies. The hyperbolic periodic orbits correspond to instability and the elliptic periodic orbits correspond to critical case which needs a special consideration.

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INFLUENCE OF PHOSPHATES AND COAL ON MANURIAL VALUE OF COMPOST

By

N. R. DHAR *and* S. P. JAISWAL

Sheila Dhar Institute of Soil Science, University of Allahabad, Allahabad

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ABSTRACT

Experiments show that phosphated composts are much better in crop production than unphosphated composts obtained from organic substances. Addition of coal powder markedly improves the crop production by composts.

The utilization of the manure was well known to the ancients. Verro (40 B.C.) emphasised the complete decay of manure before being applied to crops. Columella (50 A.D.) described the use of pits for keeping up the optimum moisture content. It appears that Romans understood the use of decayed organic materials and built manure heaps on their farms. But the scientific study and the application of composting was initiated by Hutchinson and Richards (1921) at Rothamsted and elaborated by different investigators (Howard and Wad, 1931 ; Jackson, Wad and Panse 1934 ; Smith and Coworkers 1930). Howard systematised the old procedure into a composting method known as the Indore process. The Indore process with modifications has been used widely (Howard, 1933 ; Acharya, 1950 ; Scharff, 1940, Wilson 1948). Lady Louise E. Howard (1961) has stated that in sunny climates the fixation of atmospheric nitrogen is a most advantageous factor and Prof. Dhar's researches on this point have opened up new possibilities of the greater importance.

The unbalanced character of the stabble manure as a fertilizer has long been recognized. It contains too much of nitrogen in proportion to its P_2O_5 content. Dhar *et al* (1955), Doring (1955), and Berezoa *et al* (1956) observed that the addition of phosphate to the plant material undergoing decomposition increases its nitrogen content. Dhar *et al* (1952, 1953) emphasized the fertilizing value of coal because of its richness in plant nutrients and its ability to reduce nitrogen loss from the soil.

The present investigations were undertaken to study systematically the influence of phosphates and coal both in combination as well as separately on the composting of organic matter. The fertilizing value of the resulting composts was determined by conducting pot culture experiments with paddy and wheat crops.

MATERIALS AND METHODS

Composting experiment.—The mixtute of leaves to be composted was divided into nine lots and stored in the form of heaps. The phosphate in the form of Tata and German basic slags was added at the rate of 0·5% P_2O_5 and 1·0% P_2O_5. Wherever required requisite amount of coal dust *i.e.* 1/10th of the total weight of the organic matter was also added. The materials were mixed well and water was slowly added to obtain proper moisture content. The whole mass was stirred twice per week in the first and the second month and twice per month in the succeeding period. Sufficient moisture level was maintained throughout the period of composting.

After definite intervals, composite samples were taken out, powdered, sieved and analysed on oven dry basis. Total carbon was determined by the method of Robinson, Mcleans and Williams (1929) and total nitrogen by Kjeldahl salicylic acid reduction method (John-Brooks 1936). Ammoniacal and nitrate nitrogen were determined by Olsen's method as modified by Richardson (Piper 1942), and available phosphate by Dyer's method (1934).

Pot culture experiment—Each pot was filled with 10 lbs. of soil passed through 2 mm. sieve. Sieved soil was thoroughly mixed with the required amount of compost or basic slag as the case might be. Wheat was sown on 2nd October, 1962 and harvested on 25th March, 1963. Transplanting of paddy was carried on 26th July, 1963 without any further treatment. Paddy was harvested on 28th October, 1963. Throughout the experiment, proper moisture level was maintained for growth of the crops.

Details of the expriment

Number of treatments	=	14
Number of replication	=	4
Total number of pots	=	56
Number of seedlings or seeds in each pot	=	8

The composts were mixed at the rate of 10 tons per acre

Abbreviations used :—T. B. S. for Tata Basic Slag
G. B. S. for German Basic Slag

RESULTS AND DISCUSSION

Analysis of the leaves used in composting experiment

Loss of ignition	88·5832	CaO	0·9385
Ash	11·4168	K_2O	0·9923
HCl insoluble	4·5265	MgO	0·6312
Sesquioxide	3·3812	Total carbon	38·1562
Fe_2O_3	1·5201	Total nitrogen	0·7285
P_2O_5	0·7639	C/N ratio	52·3

Analysis of the Soil

Loss on ignition	3·9628	K_2O	1·0489
Ash	96·0372	MgO	1·0203
HCl insoluble	81·6210	Total carbon	0·4903
Sesquioxide	10·0573	Total nitrogen	0 0485
Fe_2O_3	3·9800	NH_3–N	0·0022
P_2O_5	0·1043		
Available P_2O_5 (1% citric acid solution)	0·0201	NO_3–N	0·0039
		C/N ratio	10·1
CaO	1·4256		

Analysis of basic slag

	Tata basic slag %	German basic slag %
SiO_2	33·64	22·75
Fe_2O_3	6·95	16·20
K_2O	0·907	0·9525
CaO	30·3670	33·55
MgO	2·2875	5·505
Total P_2O_5	7·908	17·9050
Available P_2O_5	4·061	9·3969

TABLE 1

Temp. variation 25–40°C   Leaves + Moisture (in heaps)   Sunlight

Period of Exposure in days	Total–C%	Total–N%	NH_3-N%	NO_3-N%	Available P_2O_5%	C/N
0	38·1562	0·7285	–	–	–	52·3
50	29·7825	0·9490	0·0387	0·0562	0 0713	31·3
100	22·6908	1·0846	0·0554	0·0882	0·1135	20·9
150	17·8997	1·0299	0·0513	0·0820	0·1355	17·3
200	13·6352	0·9848	0·0470	0·0757	0·1494	13·9
250	11·5339	0·9444	0·0428	0·0661	0·1601	12·1

TABLE 2

Temp. variation 28–51·5°C   Leaves + 0·5% P_2O_5 as T.B.S.   Sunlight

0	36·6880	0·7285	–	–	–	52·3
50	26·0056	1·1248	0·0575	0·0808	0·2043	23·1
100	18·2072	1·4132	0·0916	0·1598	0·2694	12·8
150	12·4258	1·3567	0·0831	0·1497	0·3081	9·1
200	9·2257	1·3158	0·0791	0·1359	0·3262	7·0
250	7·6379	1·2631	0·0722	0·1198	0·3331	6·0

TABLE 3

Temp. variation 28–50°C   Leaves+Coal+0·5% P_2O_5 as T.B.S.   Sunlight

0	40·5647	0·8329	–	–	–	48·7
50	30·7293	1·2405	0·0501	0·0693	0·1839	24·7
100	23·0245	1·5679	0·0845	0·1365	0·2417	14·6
150	17·0113	1·5862	0·0892	0·1446	0·2950	10·7
200	13·1025	1·5510	0·0830	0·1339	0·3121	8·4
250	10·3165	1·5003	0·0771	0·1224	0·3198	6·8

TABLE 4

Temp. variation 28·5–53°C Leaves + 0·5% P_2O_5 as G.B.S. Sunlight

Period of Exposure in days	Total-C%	Total-N%	NH_3-N%	NO_3-N%	Available P_2O_5%	C/N
0	37·1167	0·7096	–	–	–	52·3
50	26·0204	1·1499	0·0641	0·0801	0 2161	22·6
100	18·9012	1·4399	0·1031	0·1561	0·2785	13·1
150	12·3263	1·3723	0·0935	0·1468	0·3162	8·9
200	9·2214	1·3312	0·0846	0·1374	0·3363	6·9
250	7·5643	1·2775	0·0726	0·1201	0·3399	5·9

TABLE 5

Temp. variation 28–51°C Leaves+Coal+0·5% P_2O_5 as G.B.S. Sunlight

0	41·0264	0·8420	–	–	–	48 7
50	31·0816	1·2528	0·0524	0·0701	0·1173	24·1
100	23·2014	1·5789	0·0853	0·1401	0·2525	14·6
150	17·1219	1·5990	0·0966	0·1451	0·2969	10·7
200	13·1068	1·5641	0·0891	0·1380	0·3184	8·3
250	10 2247	1·5114	0·0782	0·1264	0·3275	6·7

TABLE 6

Temp. variation 29–35·5°C Leaves+1% P_2O_5 as T.B.S. Sunlight

0	34·8744	0·6617	–	–	–	52·3
50	24·7214	1·2178	0·0580	0·0819	0·2314	20·3
100	16·1258	1·4318	0·0943	0·1621	0·2786	11·2
150	11·0154	1·3706	0·0864	0·1532	0·3308	8·4
200	8·3138	1·3291	0·0809	0·1401	0·3614	6·6
250	6·8224	1·2805	0·0756	0·1208	0·3766	5·7

TABLE 7

Temp. variation 29–51°C Leaves+Coal+1% P_2O_5 as T.B S. Sunlight

0	38·6321	0·7938	–	–	–	48·7
50	29·1455	1·2514	0·0526	0·0708	0·1962	23·3
100	21·8426	1·5803	0·0872	0·1391	0·2532	13·8
150	15·9624	1·5987	0·0908	0·1461	0·3028	10·3
200	12 0836	1·5623	0·0863	0·1388	0·3350	7·7
250	9·5458	1·5141	0·0177	0·1286	0·3427	6·3

TABLE 8

Temp. variation 29–45·5°C Leaves+1% P_2O_5 as G.B.S. Sunlight

Period of Exposure in days	Total–C%	Total–N%	NH_3–N%	NO_3–N%	Available P_2O_5%	C/N
0	36·1269	0·6898	–	–	–	52·3
50	24·9824	1·2436	0·0665	0·0908	0·2448	20·1
100	16·0108	1·4454	0·0965	0·1701	0·2904	11·1
150	10·9095	1·3883	0·0882	0·1524	0·3584	7·9
200	8·1883	1·3401	0·0811	0·1408	0·3802	6·1
250	6·6745	1·2954	0·0176	0·1216	0·3923	5·2

TABLE 9

Temp. variation 28–52·5°C Leaves+Coal+1% P_2O_5 as G.B.S. Sunlight

0	40·7972	0·8377	–	–	–	48·7
50	29·0118	1·2677	0·0537	0·0723	0·2007	22·9
100	21·6162	1·5919	0·0887	0·1408	0·2594	13·5
150	15·8238	1·6092	0·0933	0·1479	0·3143	9·8
200	11·7974	1·5770	0·0899	0·1402	0·3484	7·5
250	9·4047	1·5213	0 0884	0·1301	0·3588	6·1

The foregoing results show clearly that there is a considerable oxidation of carbonaceous compounds when leaves are allowed to undergo slow oxidation in air. Moreover, when slags are incorporated the velocity of oxidation is appreciably increased. This may be due to the alkaline nature of the slags and that alkalinity favours oxidation is a well known fact. During composting process, a progressive increase in nitrogen concentration has also been observed. The increase in nitrogen is more pronounced in systems containing slag. This is due to the fact that more nitrogenous compounds are created from the air per unit weight of the carbon oxidised in presence of basic slags than in their absence. Loss of organic matter and increase in nitrogen during the composting of organic materials has been reported by a number of workers (Howard and Wad 1931, Gotaas and Coworkers 1953, Dhar *et al* 1951).

The reason for considerable nitrogen fixation in presence of phosphate is not far to seek. Dhar has explained it from the view point that during the process of ammonification and nitrification of proteins a highly unstable intermediate product *i.e.* ammonium nitrite is formed (Proteins → amino acid $\xrightarrow{+O_2}$ ammonium compounds $\xrightarrow{+O_2} NO_2 \xrightarrow{+O_2} NO_3$). It readily decomposes forming water and nitrogen gas with marked evolution of heat. This, alongwith nitrogen fixation and formation of protein caused by the absorption of the energy obtained from the oxidation of organic materials in the compost, ammonification and nitrification of the

nitrogenous compound formed and present in the system which opposes the increase of proteins in the composts, takes place and thus the amount of proteins remaining in the composts tends to decrease. But in the presence of phosphates in the system more or less stable phosphoproteins are formed by the combination of proteins and phosphates (Dhar and Ghosh, 1956). These compounds seem to resist ammonfication, nitrification and loss of nitrogen better than phosphates alone. Moreover, basic slags also supply calcium and magnesium ions in the system during the decomposition of organic matter to form calcium and magnesium nitrite which are more stable than ammonium nitrite with the result nitrogen of the system is conserved.

Pot culture experiment

Sl. No.	Treatments	Means			
		Yield of wheat grain in gm	Yield of wheat straw in gm	Yield of paddy grain in gm	Yield of paddy straw in gm
1.	Soil alone (control)	10·25	21·25	16·50	28·00
2.	Soil+compost (Unphosphated)	21·25	34·75	35·25	55·75
3.	Soil + G.B.S. (Same amount of P_2O_5 as in compost containing 1% P_2O_5 as G.B.S.)	13·75	26·00	22·25	37·50
4.	Soil + G B S. (Same amount of P_2O_5 as in compost containing 0 5% P_2O_5 as G.B.S.)	12·75	22·50	19·75	39·00
5.	Soil + T.B.S. (Same amount of P_2O_5 as in compost containing 1% P_2O_5 as T.B.S.)	13·75	25·00	21·25	35·25
6.	Soil + T.B.S. (Same amount of P_2O_5 as in compost containing 0·5% P_2O_5 as T.B.S.)	12·00	22·00	19·00	31·00
7.	Soil + compost containing 1% P_2O_5 as G.B.S.	32·75	52·00	53·50	82·75
8.	Soil + compost containing 0·5% P_2O_5 as G.B.S.	30·2	46·00	48 00	75·00
9.	Soil + compost containing 1% P_2O_5 as T.B.S.	30·75	49·00	50·50	79·50
10.	Soil + compost containing 0·5% P_2O_5 as T.B S.	28·25	45·25	47·00	73·25
11.	Soil + compost containing Coal + 1% P_2O_5 as G.B.S.	42·75	71·00	67·50	104·25
12.	Soil + compost containing Coal + 0·5% P_2O_5 as G.B.S.	37·50	61·00	62·00	97·75
13.	Soil + compost containing Coal + 1% P_2O_5 as T.B.S.	36·00	64·00	64 00	101·25
14.	Soil + compost containing Coal + 0·5% P_2O_5 as T.B.S.	36·00	58·00	58·25	94·21
C. D. at 5% level		2·36	10·11	12·15	6·71

A close examination of the results shows that when coal is mixed with the leaves undergoing decomposition and oxidation in heaps, the loss of nitrogen with the lapse of time is reduced and coal compost is richer in nitrogen than the compost prepared in absence of coal. The decrease in nitrogen loss can be explained on the basis that coal acts as a negative catalyst. It is well known that the carbohydrates preserve body proteins from undergoing oxidation and it is just likely that carbonaceous matter present in the coal may act in such a way as to protect the protein of the nitrogenous compounds of the compost from oxidation. It appears that coal may also absorb some ammonia and thus retard its loss as coal and lignite possess great absorbing capacity for gases. Dhar and Agarwal (1952) observed that carbonaceous material in the form of coal when added to the soil with ammonium sulphate retards the loss of nitrogen.

From our experiments recorded in foregoing pages it is clear that for the preparation of compost both basic slags as well as coal are useful as the compost prepared with these contain greater amount of total nitrogen, ammoniacal nitrogen and nitrate nitrogen.

From the yield data it is evident that by the application of different composts to the soil in pots much better crop yields are obtained than in the control. The efficiency of different treatments in increasing the yield of wheat and paddy crops is in the following order :—

Compost containing 1% P_2O_5 as G.B.S. and coal > Compost containing 1% P_2O_5 as T.B.S. and coal > Compost containing 0·5% P_2O_5 as G.B.S. and coal > Compost containing 0·5% P_2O_5 as T.B.S. and coal > Compost containing 1% P_2O_5 as G.B.S. > Compost containing 1% P_2O_5 as T.B.S. > Compost containing 0·5% P_2O_5 as G.B.S. > Compost containing 0·5% P_2O_5 as T.B.S. > Unphosphated compost > 1% P_2O_5 as G.B.S. > 1% P_2O_5 as T.B.S. > 0·5% P_2O_5 as G.B.S. > 0·5% P_2O_5 as T.B.S. > conrol.

These experimental results show that the composts which have been prepared by the incorporation of phosphates have given much better yields as compared to unphosphated compost. This significant difference in yield seems to be due to the presence of higher amount of total nitrogen, available nitrogen, available phosphate and other plant nutrients. Copeland and Merckle (1942), Åslander (1953), Dalton, Russell and Sieling (1952) attributed this beneficial effect to high nitrogen content in the compost. These results also show that the additon of the same amount of P_2O_5 as present in the phosphated compost produces much lower yield than the phosphated compost. It appears, therefore, that the mixture of compost and phosphatic material possesses several advantages over an application of phosphates alone to the soil.

The yield data also shows that the phosphated coal mixed compost is better than the phosphated compost. This may be due to the presence of higher amount of total and available nitrogen in the former than in the latter. Dhar *et al* (1953) carried out field trials and found bituminous coal very favourable for growing both paddy and wheat.

Thus it is obvious from the experimental results recorded in foregoing pages that the composting of organic materials if carried on in presence of phosphates and coal will be a fruitful proposition for increasing the manurial value of the composts.

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INFLUENCE OF LIGHT PHOSPHATE AND ORGANIC MATTER INOCULATED WITH ANABAENA NAVICULOIDES AND CHLORELLA PYPRENOIDOSA ON NITROGEN FIXATION IN SOIL

By

N. R. DHAR *and* S. P. JAISWAL

Sheila Dhar Institute of Soil Science, University of Allahabad, Allahabad

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ABSTRACT

Experimental results have been recorded showing that the addition of algae does not increase markedly the thermal and photo-chemical nitrogen fixation in the slow oxidation of organic matter. Phosphates markedly increase this nitrogen fixation.

Algae create carbonaceous compounds in the soil in small amounts and check nitrogen loss from soils in the nitrification of nitrogenous compounds.

In the recent years emphasis is being laid on the part played by blue green algae in maintaining soil fertility (De, 1939 ; Fogg, 1942 ; 1951 and 1952 ; Watanabe *et al*, 1951). De and Sulaiman (1942) in their experiments with rice plants observed nitrogen fixation in pots inoculated with algae. Dhar and Gupta (1954) have experimentally shown that algal contribution in enriching soils is very small as compared to that of the organic matter undergoing slow oxidation. De and Mandal (1956) obtained about 13-44 lbs. of fixed nitrogen in unfertilized water logged soils inoculated with blue green algae.

Organic matter seems to enhance the growth of algae. Bristol Reach (1932) at Rothamsted found that generous dressing of Farm-yard manure increased the number of blue-green algae as compared with the unmanured soils. Pearsal (1932) observed that blue-green algae are found in abundance in fresh water especially with high content of dissolved organic matter. Phosphate and lime have also been reported to be helpful in increasing the growth and activity of algae (De and Sulaiman, 1942 : Okuda and Yamaguchi, 1952).

The present investigations were undertaken to study systematically the influence of algae on nitrogen fixation in soil in conjunction with energy material in the form of Wheat straw and Phosphates in the form of Tata and German basic slags. In view of their importance in plant nutrition the determination of available nitrogen, available phosphate and free amino-acids was also undertaken.

METHODS AND MATERIALS

The following media have been used for growing Anabaena naviculoides and Chlorella pyrenoidosa :

Medium for Anabaena		Medium for Chlorella	
KNO_3	0·2 gm.	NH_4NO_3	0·2 gm.
K_2HPO_4	0·2 gm.	K_2HPO_4	0 2 gm.
$MgSO_4 \cdot 7H_2O$	0 2 gm.	$MgSO_4 \cdot 7H_2O$	0·2 gm.
$CaCl_2$	0·1 gm.	$CaCl_2$	0·1 gm.
$FeCl_3$ (1% solution)	2 drops	$FeCl_3$ (1% solution)	2 drops.
Distilled water	1000 cc.	Distilled water	1000 cc.

All the constituents *viz.* wheat straw, soil and basic slags used in these experiments were screened through 100 mesh sieve. 100 gms. of air dried soil were taken in clean enamelled dishes to which required amount of organic material

and phosphates were added and thoroughly mixed. Wherever required the soil in dishes were inoculated with Anabaena naviculoides and Chlorella pyrenoidosa, already grown in the given media. Two identical sets of experiments were started simultaneously. One was exposed to light from a 100 watt bulb and other was kept covered with thick black cloth adjacent to the exposed sets. Throughout the experiment moisture content was maintained at 40% level.

After definite intervals of time composite soil samples were taken out and analysed on oven dry basis for all the constituents. Total carbon was determined by the method of Robinson, Mclean and Williams (1929) and total nitrogen by modified Kjeldahl method (John-Brooks) to include nitrate nitrogen. Available phosphate was determined by Dyer's method (1934). Ammonical and nitrate nitrogen by Olsen's method modified by Richardson (Piper 1942). Free aminoacids in the sample were extracted by the method of Payne *et al* (1956). Paper Partition Chromatographic techniques as developed by Giri *et al* (1952) and Consden *et al* (1944) were employed for the identification of amino acids. Quantitative estimations were carried out according to the colorimetric method of Harding and Mclean (1916).

The following abbreviations have been used for different amino-acids in this paper.

Gly—Glycine	His—Hissidine
Al—Alanine	Asp—Aspartic acid
Va—Valine	As—Asparagine
Ly—Lycine	Glu—Glutamic acid
Ar—Arginine	Leu—Leucine
Pro—Proline	Threo—Threonine
	Se—serine

RESULTS

Analysis of the soil

Moisture	2·57%	C/N	12·1
Loss on ignition	4·2410%	NH_3-N	0·0050%
HCl insoluble	73·4662%	NO_3-N	0·0079%
Sesquioxide	12·4012%	Exchangeable Ca (Na Cl method)	9·8 m.e./100 gms.
Fe_2O_3	4·3897%		
CaO	2 8616%	Available P_2O_3	0·0411%
MgO	1·8372%	pH	7·5
P_2O_5	0·1903%	Total bacterial count	11·6 millions/gms.
K_2O	0·8610%	Azotobacter number	1·8 millions/gm.
Totol carbon	1·4614%	Fungus count	51·000/gm.
Total nitrogen	0·1202%		

Analysis of wheat straw

Total carbon	36·6124%
Total nitrogen	0·6024%
C/N	60·7

Analysis of basic slags

	Tata basic slag%	German basic slag%
SiO_2	33·64	22·75
Fe_2O_3	6 95	16·20
K_2O	0·907	0·9525
CaO	30 3670	33·55
MgO	2 2875	5 505
Total P_2O_5	7·908	17·9050
Available P_2O_5	4·061	9·3969

TABLE I

Average temperature 28°C

Period of exposure in days	Total C (gm.)	Total N (gm.)	NH_3—N (mgm.)	NO_3—N (mgm.)	Available P_2O_5 %	Amino acids identified (Chromatographically)	Amino acid content with respect to glycine (mgm.)	Efficiency
					LIGHT			
0	1·4614	0·1202	5·0	7·9	0·0411	Al,Va,Glu.	..	..
60	1·3808	0·1215	5·2	8·3	0·0416	Al,Va,Glu.Ly.	..	16·1
120	1·3204	0·1224	5·3	8·5	0·0419	Al,Va,Glu.Ly.	0·0308	15·6
180	1·2847	0·1227	5·3	8·9	0·0422	Al,Va,Glu.Ly.	0·0345	14·1
240	1·2553	0·1219	5·0	8·7	0·0423	Al,Va,Glu.LyandAr.	0·0331	..
					DARK			
0	1·4614	0·1202	5 0	7·9	0·0411	Al,Va,Glu.	..	..
60	1·4012	0·1207	5·2	8·0	0·0415	Al,Va,Glu.	..	8·3
120	1·3510	0·1211	5·3	8·2	0·0418	Al,Va,Glu.Ly.	..	8·1
180	1·3315	0·1212	5·3	8·3	0·0419	Al,Va,Glu.Ly.	0·0267	7·6
240	1·3171	0·1207	5·1	8·2	0·0420	Al,Va,Glu,Ly.	0·0272	..

TABLE II

Average temperature 28°C

Period of exposure in days	Total C (gm.)	Total N (gm.)	NH_3—N (mgm.)	NO_3—N (mgm.)	Available P_2O_5 %	Amino acids identified (Chromatographically)	Amino acid content with respect to glycine (mgm.)	Efficiency
				Soil + Chlorella pyrenoidosa				
				LIGHT				
0	1·4614	0·1202	5·0	7·9	0·0411	Al,Va,Glu.	..	..
60	1·4017	0·1216	5·2	8·1	0·0416	Al,Va,Glu.Ly,Asp.	..	23·4
120	1·3544	0·1226	5·2	8·4	0·0420	Al,Va,Glu,Ly,Asp.	0·0315	22·4
180	1·3310	0·1229	5·3	8·6	0·0423	Al,Va,Glu,Ly,Asp,Ar.	0·0358	20·7
240	1·3165	0·1222	5·1	8·4	0·0425	Al,Va,Glu,Ly,Ar,Asp.	0·0342	..
				DARK				
0	1·4614	0 1202	5·0	7·9	0·0411	Al,Va,Glu.	..	..
60	1·4024	0·1207	5·2	8·0	0·0415	Al,Va,Glu.	..	8·4
120	1·3514	0·1211	5·2	8·3	0·0417	Al,Va,Glu,Ly.	..	8·1
180	1·3324	0·1212	5·3	8·3	0·0419	Al,Va,Glu,Ly.	0·0269	7·7
240	1·3173	0·1207	5·2	8·1	0·0421	Al,Va,Glu,Ly.	0 0271	..

TABLE III

Average temperature 28°C

Period of exposure in days	Total C (gm.)	Total N (gm)	NH_3—N. (mgm.)	NO_3—N (mgm.)	Available P_2O_5 %	Amino acids identified (Chromatographically)	Amino acid content with respect to glycine (mgm.)	Efficiency
Soil + Anabaena naviculoides								
LIGHT								
0	1·4614	0·1202	5·0	7·9	0·0441	Al,Va,Glu.	..	..
60	1·4019	0·1219	5·3	8·1	0·0417	Al,Va,Glu,Ly,Asp.	..	28·5
120	1·3564	0·1230	5·3	8·4	0·0421	Al,Va,Glu,Ly,Asp, Ar.	0·0331	26·6
180	1·3513	0·1234	5·4	8·6	0·0425	Al,Va,Glu,Ly,Asp,Ar.	0·0380	24·5
240	1·3163	0·1227	5·2	8·4	0·0427	Glu,Ly,Al, Va,Asp,Ar.	0·0362	..
DARK								
0	1·4614	0·1202	5·0	7·9	0·0411	Al,Va,Glu.	..	..
60	1·4022	0·1207	5·2	8·0	0·0415	Al,Va,Glu.	..	8·4
120	1·3516	0·1211	5·3	8·2	0·0418	Al,Va,Glu,Ly.	..	8·2
180	1·3523	0·1212	5·3	8·3	0·0419	Al,Va,Glu,Ly.	0·0271	7·7
240	1·3171	0·1208	5·2	8·1	0·0420	Al,Va,Glu,Ly.	0·0275	..

TABLE IV

Average temperature 28°C

Period of exposure in days	Total C (gm)	Total N (gm.)	NH_3—N. (mgm.)	NO_3—N (mgm.)	Available P_2O_5 %	Amino acids identified (Chromatographically)	Amino acid content with respect to glycine (mgm.)	Efficiency
Soil + 1% Carbon as Wheat straw								
LIGHT								
0	2·4614	0·1367	5·0	7·9	0·0400	Al,Va,Glu.	..	..
60	2·1782	0·1454	5·8	9·1	0·0441	Al,Va,Glu,Ly, Asp,Lcu.	1·7832	30·7
120	1·9593	0·1518	6·7	10·9	0·0469	Al,Va,Glu,Ly,Asp,Threo, Leu.	2·4428	30·0
180	1·7883	0·1562	7·5	13·6	0·0487	Al,Va,Glu,Ly,Asp,Threo, Leu,Ar.	2·7485	28·9
240	1·7087	0·1551	7·3	13·4	0·0498	Al,Va,Glu,Ly,Asp,Threo, Leu,Ar.	2·6572	..
DARK								
0	2·4614	0·1367	5·0	7·9	0·0400	Al,Va,Glu.	..	..
60	2·2294	0·1405	5·4	8·4	0·0434	Al,Va,Glu,Asp,Ly.	1·0943	16·3
120	2·0554	0·1432	6·1	9·7	0·0498	Al,Va,Glu,Ly,Asp,Threo,	1·5284	16·0
180	1·9124	0 1492	6·7	11·4	0·0475	Al,Va,Glu,Ly,Asp,Threo, Leu.	1·9163	15·4
240	1·8432	0·1447	6·5	11·4	0·0482	Al,Va,Glu,Ly,Asp,Threo, Leu.	1·8775	..

TABLE V

Average temperature 28°C

Period of exposure in days	Total C (gm.)	Total N (gm.)	NH_3—N (mgm.)	NO_3—N (mgm.)	Available P_2O_5 %	Amino acids identified (Chromatographically)	Amino acid content with respect to glycine (mgm.)	Efficiency
Soil + 1% Wheat straw + Chlorella Pyrenoidosa								
LIGHT								
0	2·4614	0·1367	5·0	7·9	0·0400	Al,Va,Glu.	. .	. .
60	2·2380	0 1456	5·8	8 9	0·0442	Al.Va,Glu,Asp,Ly,Threo, Leu.	1·8128	39·9
120	2·0451	0·1522	6·5	10·7	0·0472	Al,Va,Asp,Ly,Ar,Glu, Threo,Leu.	2·4859	37·2
180	1·8907	0·1566	7·4	13·4	0·0491	Al,Va,Asp,Ly,Ar,Glu Threo,Leu,Pro.	2·7966	34·8
240	1·8286	0·1556	7·2	13·1	0·0505	Al,Va,Asp,Ly,Ar,Glu, Threo,Leu,Pro.	2·7026	. .
DARK								
0	2·4614	0·1367	5·0	7·9	0·0400	Al,Va,Glu.	. .	1·0
60	2·2299	0·1405	5·4	8·3	0·0435	Al,Va,Glu,Ly,Asp.	1·0454	16·4
120	2·0555	0·1433	6·1	9·7	0·0458	Al,Va,Glu,Ly,Asp,Threo.	1·5296	16.2
180	1·9129	0·1452	6·8	11·4	0·0474	Al,Va,Glu,Asp,Ly,Threo.	1·9164	15·5
240	1·8434	0·1447	6·6	11·3	0·0482	Al,Va,Glu,Asp,Ly,Threo, Leu.	1·8773	. .

TABLE VI

Average temperature 28°C

Period of exposure in days	Total C (gm.)	Total N (gm.)	NH_3—N (mgm.)	NO_3—N (mgm.)	Available P_2O_5 %	Amino acids identified (Chromatographically)	Amino acid content with respect to glycine (mgm.)	Efficiency
Soil + 1%C as Wheat straw + Anabaena naviculoides								
LIGHT								
0	2·4614	0·1367	5·0	7·9	0·0400	Al, Va, Glu.	..	..
60	2·2386	0·1467	5·8	8·8	0·0422	Al,Va,Glu,Asp,Ly,Threo, Leu.	2·0485	46·2
120	2·0541	0·1538	6·6	10·6	0·0474	Al,Va,Asp,Ly,Glu,Threo, Leu, Pro.	2·7316	41·9
180	1·8995	0·1587	7·4	13·5	0·0493	Al,Va,Asp,Ly,Ar.Glu, Threu,Leo,Pro.	2·9531	39·1
240	1·8303	0·1576	7·3	13·3	0·0508	Al,Va,Ly,Glu,Ar,Threo, Asp,Leu,Pro.	2·8065	..
DARK								
0	2·4614	0·1367	5·0	7·9	0·0400	Ala,Va,Glu.	..	..
60	2·2298	0·1405	5·4	8·4	0·0434	Al,Va,Glu,Asp,Ly.	1·0951	15·4
120	2·0556	0·1433	6·1	9·7	0·0458	Al,Va,Glu,Ly,Asp,Threo.	1·5295	16·2
180	1·9128	0·1452	6·8	11·4	0·0474	Al,Va,Glu,Ly,Asp,Threo, Leu.	1·9162	15·4
240	1·8435	0·1448	6·6	11·4	0·0483	Al,Va,Glu,Ly,Asp,Threo, Leu.	1·9772	..

TABLE VII

Average temperature 28°C

Period of exposure in days	Total C (gm.)	Total N (gm.)	NH_3—N (mgm.)	NO_3—N (mgm.)	Available P_2O_5 %	Amino acids identified (Chromatographically)	Amino acid content with respect to glycine (mgm.)	Efficiency
Soil + 1% C as Wheat straw + 0·5% P_2O_5 as Tata basic slag								
LIGHT								
0	2·4614	0·1367	5·0	7·9	0·2729	Al,Va,Glu.	..	..
60	2·1566	0·1501	6·3	10·1	0·2922	Ly,Al,Va,Glu,Asp,Threo, Pro.	2·5573	44·1
120	1·9293	0·1600	7·9	15·7	0·3057	Ly,Va,Asp,Glu,Threo,Leu, Ar,His.	3·7063	43·8
180	1·7474	0·1672	9·1	17·5	0·3141	Ly,Va,Glu,Asp,Ar,Threo, Leu,His,As.	4·4634	42·8
240	1·6586	0·1659	8·9	17·2	0·3192	Ly,Va,Asp,Ar,Glu,Threo, His,Leu,As.	4·3040	..
DARK								
0	2·4614	0·1367	5·0	7·9	0·2729	Al,Va,Glu.	..	..
60	2·2053	0·1426	5·9	9·2	0·2883	Ly,Al,Va,Glu,Asp,Threo.	1·6478	23·0
120	2·0164	0·1469	7·0	11·4	0·2996	Ly,Va,Glu,Al,Threo,Leu, Ar.	2·4373	22·9
180	1·8532	0·1502	7·9	13·4	0·3070	Ly,Va,Glu,Asp,Leu, Threo,Ar,His.	2·8287	22·2
240	1·7751	0·1497	7·7	13·3	0·3117	Ly,Va,Glu,Asp,Leu, Threo,Ar,His.	2·7490	..

TABLE VIII

Average temperature 28°C

Period of exposure in days	Total C (gm.)	Total N (gm.)	NH_3—N (mgm.)	NO_3—N (mgm.)	Available P_2O_5 %	Amino acids identified (Chromatographically)	Amino acid content with respect to glycine(mgm.)	Efficiency
Soil + 1% C as Wheat straw 0·5% P_2O_5 as Tata basic slag + Chlorella pyrenoidosa								
LIGHT								
0	2·4614	0·1367	5·0	7·9	0·2729	Al,Va,Glu.	..	..
60	2·2051	0·1505	6·2	9·9	0·2927	Ly,Asp,Glu,Va,Al,As, Threo,Leu.	2·5988	53·8
120	2·0031	0·1604	7·7	13·5	0·3067	Ly,Asp,Glu,Va,Al,As, Threo,Pro,Ar,Leu.	3·7622	51·7
180	1·8394	0·1677	8·9	17·2	0·3153	Ly Asp,Glu,Va,As,Threo, Pro,Leu;Ar,His.	4·5225	49·8
240	1·7626	0·1666	8·7	17·0	0·3206	Ly,Asp,Glu,Va,As,Threo, Pro,Leu,Ar,His.	4·3619	..
DARK								
0	2·4614	0·1367	5·0	7·9	0·2729	Al,Va,Glu.	..	..
60	2·2058	0·1426	5·9	9·2	0·2882	Ly,Al,Va,Glu,Asp,Threo,	1·6471	23·9
120	2·0175	0·1469	7·0	11·3	0·2996	Ly,Va,Asp,Glu,Al,Threo, Leu.	2·4379	22·9
180	1·8538	0·1502	7·9	13·3	0·3071	Ly,Va,Asp,Glu,Al,Threo, Leu,Ar,His.	2·8290	22·2
240	1·7758	0·1497	7·7	13·3	0·3119	Ly,Va,Asp,Glu,Leu,Ar, His.	2·7493	..

TABLE IX

Average temperature 28°C

Period of exposure in days	Total C (gm.)	Total N (gm.)	NH_3—N (mgm.)	NO_3—N (mgm.)	Available P_2O_5 %	Amino acids identified (Chromatographically)	Amino acid content with respect to glycine (mgm.)	Efficiency
Soil + 1% C as Wheat straw + 0·5 % P_2O_5 as Tata basic slag + Anabaena naviculoides								
LIGHT								
0	2·4614	0·1367	5·0	7·9	0·2729	Al,Va,Glu.	..	..
60	2·2031	0·1522	6·3	9·9	0·2927	Ly,Asp,Glu,Al,As,Threo, Pro,Leu,Va.	2·8613	60·4
120	2·0149	0·1626	7·8	13·5	0·3069	Ly,Asp,Glu,Va,Al,As, Threo,Pro,Ar.	4·0288	57·7
180	1·8418	0·1705	9·1	17·3	0·3154	Ly,Asp,Glu,Va,As,Threo, His,Ar,Se,Pro.	4·7296	54·5
240	1·7639	0·1692	8·9	17·0	0·3208	Ly,Asp,Glu,As,Va,His,Ar, Threo,Pro.Se.	4·5042	..
DARK								
0	2·4614	0·1367	5·0	7·9	0 2729	Al,Va,Glu.	..	..
60	2·2056	0·1426	5·9	9·1	0·2885	Ly,Al,Va,Glu,Asp,Threo.	1·6409	23·0
120	2·0172	0·1469	7·0	11·3	0·2997	Ly,Va,Asp,Glu,Al,Threo, Leu,Ar.	2·4385	22·9
180	1·8536	0·1503	7·9	13·4	0·3070	Ly,Va,Glu,Asp,Leu,Threo, Ar,His.	2·8192	22·3
240	1·7759	0·1498	7·7	13·3	0·3118	Ly,Va,Glu,Asp,Leu,Threo, Ar,His.	2·7493	..

TABLE X

Average temperature 28°C

Period of exposure in days	Total C (gm.)	Total N (gm.)	NH_3—N (mgm.)	NO_3—N (mgm.)	Available P_2O_5 %	Amino acids identified (Chromatographically)	Amino acid content with respect to glycine (mgm.)	Efficiency
Soil + 1% C as Wheat straw + 0·5% P_2O_5 as German basic slag								
LIGHT								
0	2·4614	0·1367	5·0	7·9	0·2880	Al,Va,Glu.	..	..
60	2·1556	0·1505	6·4	10·2	0·3073	Al,Va,Ly,Glu,Asp,Threo, His.	2·7131	45·2
120	1·9278	0·1607	8 0	13·9	0·3114	Ly,As,Va,Glu,Ar,Asp, Threo,His,Leu.	3·8856	45·0
180	1·7456	0·1679	9·3	17·4	0·3500	Ly,Va,Glu,As,Ar,Asp, Threo,His,Leu.	4·5673	43·7
240	1·6562	0·1665	9·0	17·2	0·3350	Al,Va,Ly,Glu,Ar,Asp, Threo,His,Leu.	4·4080	..
DARK								
0	2·4614	0·1367	5·0	7·9	0·2880	Al,Va,Glu.	..	..
60	2·2029	0·1428	5·9	9·3	0·3037	Ly,Asp,Al,Va,Glu, Threo.	1·7211	23·7
120	2·0131	0·1472	7·1	11·5	0·3152	Ly,Va,Glu,Asp,Ar,Threo, His,As,Al.	2·4178	23·5
180	1·8523	0·1506	7·9	13·5	0·3226	Ly,Va,Asp,Ar,As,Glu, Threo,His,Al.	2·9115	22·8
240	1·7739	0·1500	7 7	13·4	0·3272	Al,Va,Asp,Ar,Glu,Threo, His,As.	2·8256	..

TABLE XI

Average temperature 28°C

Period of exposure in days	Total C (gm.)	Total N (gm.)	NH_3—N (mgm.)	NO_3—N (mgm.)	Available P_2O_5 %	Amino acids identified (Chromatographically)	Amino acid content with respect to glycine (mgm.)	Efficiency
Soil + C as Wheat straw + 0·5 % P_2O_5 as German basic slag + Chlorella pyrenoidosa								
LIGHT								
0	2·4614	0·1367	5·0	7·9	0·2880	Al,Va,Glu.	. .	. .
60	2·2016	0·1509	6·3	10·0	0 3083	Al,Va,Ly,Glu,As,Asp, Threo,His.	2·7526	54·7
120	2·0032	0·1612	7·9	13·6	0·3223	Ly,As,Va,Asp,Ar,Pro, Threo,Leu,His.	3·9275	53·2
180	1·8345	0·1684	9·2	17·1	0 3513	Ly,As,Va,Asp,Ar,Pro, Threo,Leu,His,Se.	4·6158	50·5
240	1·7601	0·1671	8·9	16·8	0·3364	Ly,As,Va,Asp,Ar,Pro, Threo,Leu,His,Se.	4·4535	. .
DARK								
0	2·4614	0·1367	5·0	7·9	0·2880	Al,Va,Glu.	. .	. .
60	2·2133	0·1429	5·9	9·2	0·3036	Ly,Asp,Al,Va,Glu,Threo.	1·7220	24·0
120	2·0144	0·1472	7·1	11·5	0·3151	Asp,Ly,Va,Glu,Ar,Threo, His,As,Al.	2·4279	23·5
180	1·8532	0·1506	8 0	13·4	0·3226	Ly,Va,Glu,Asp,Ar,Threo, His,As,Al.	2·9121	22·9
240	1·7748	0 1500	7·8	13·3	0·3271	Ly,Va,Glu,Asp,Ar,Threo, His,As.	2·8259	. .

TABLE XII

Average temperature 28°C

Period of exposure in days	Total C (gm)	Total N (gm.)	NH_3—N (mgm.)	NO_3—N (mgm.)	Available P_2O_5 %	Amino acids identified (Chromatographically)	Amino acid content with respect to glycine (mgm.)	Efficiency
Soil + 1% C as Wheat Straw + 0·5% P_2O_5 as German basic slag + Anabaena naviculoides								
LIGHT								
0	2·4614	0·1367	5·0	7·9	0·2880	Al,Va,Glu.	..	..
60	2·2021	0·1527	6·4	10·0	0·3084	Al,Va,Ly,Glu,Asp,Threo, His,As.	3·0176	61·3
120	2·0109	0·1632	8·0	13·7	0·3226	Ly,As,Va,Asp,Pro,Threo, Ar,Leu,His.	4·2090	58·8
180	1·8407	0·1712	9·3	17·2	0·3315	Ly,As,Va,Asp,Pro,Threo, Ar,Leu,His,Se.	4·8332	55·5
240	1·7622	0·1698	9·1	17·0	0·3567	Ly,As,Va,Asp,Pro,Ar, Threo,Leu,His,Se.	4·6081	..
DARK								
0	2·4614	0·1367	5·0	7·9	0 2880	Al,Va,Glu.	..	..
60	2·2035	0·1429	5·9	9·3	0·3036	Ly,Asp,Al,Va,Glu,Threo.	1·7210	24·0
120	2·0145	0·1472	7·2	11·4	0·3050	Ly,Va,Glu,Ar,Asp,Threo, His,As,Al.	2·4186	23·5
180	1·8531	0·1505	7·9	13·5	0·3225	Ly,Va,Glu,Asp,Ar.Threo, His,As.	2·9119	23·0
240	1·7746	0·1501	7·7	13·4	0·3272	Ly,Va,Glu,Asp,Ar,Threo, His,As.	2·8298	..

DISCUSSION

The results recorded in table I show that when soil humus is allowed to undergo slow oxidation in presence of air, there is a small increase in nitrogen of the system. Similarly Dhar (1952) reported a small increase in nitrogen without added organic matter in Indian soils rich in calcium phosphate. Remy (1909), Mockeridge (1917) and deRossi (1932) etc. found gain in nitrogen by inoculating soils several months. It is further observed that when soil is inoculated with algae, there is some saving of carbon and slight increase in Nitrogen. The increase in nitrogen is greater with Anabaena naviculoides than with Chlorella pyrenoidosa. This may be due to the fact that Anabaena is capable of fixing small amount of nitrogen autotrophically.

A perusal of the preceding results show that wheat straw when added to the soil is slowly oxidised in contact with air and there is a considerable increase in the total nitrogen content of the system. This clearly shows that the energey released during the process of carbon oxidation is actually utilised in fixing atmospheric nitrogen. It is observed that when the syrtem is illuminated, the rate of oxidation of carbonaceous compounds and the increase in the total nitrogen are markedly enhanced. Moreover, the efficiency of nitrogen fixation is always in a ratio of approximately 2 : 1 in light/dark showing that the light energy absorbed by the system is also utilised in accomplishing the endothermal reaction of ammonia formation. This efficiency is further increased when phosphate in the form of basic slags are incorporated alongwith organic material.

A close examination of the experimental results reveals that in presence of energy material, there is a marked saving of carbon and greater increase in nitrogen in the sets inoculated with algae. It seems, therefore, that organic matter activates these organisms. De and Mandal (1956) reported greater nitrogen fixation by blue green algae in presence of crop than in its absence. They attributed this stimulating effect to the increased supply of carbon dioxide resulting from the respiration and decomposition of roots of higher plants. Furthermore, in the presence of phosphates the growth and activity of these algae, seem to be further enhanced. This is in agreement with the findings of Okuda and Yamaguchi (1952).

The efficiency of nitrogen fixation *i.e.* the amount of nitrogen fixed in mgm. per gram of carbon oxidised is greater in the prescnce of algae than in their absence. Moreover, the efficiency of nitrogen fixation is always greater in the case of Anabaena than in the case of Chlorella. The increase in efficiency in algae sets seems to be due to the addition of organic matter by the growth of algae and its consequent oxidation and fixation of nitrogen. Since Anabaena is endowed with the power of fixing nitrogen autotrophically, its efficiency is greater than that of Chlorella. It is further observed that efficiency of nitrogen fixation in the algae sets falls off with time. If shows that algal growth and activity is more pronounced in the beginning than in the end.

The experimental results show that the available phosphate of the system increases with the oxidation of carbon and it is greatly enhanced when phosphates are added. In the algal sets the increase in the availability of phosphates is slightly greater in the systems without algae. Besides the increase in availability due to production of more carbonic acid and organic acids, the organic phosphorous of the nucleic acid and nucleotides etc. of the algal material may be made available by dephosphorylation (Thompson and Black 1950).

A perusal of the prceeding results also shows that with the increase in available nitrogen there is an increase in number and amount of free aminoacids in the system. The number and amount of free aminoacids produced is greater

in the sets exposed to light than in those kept in the dark. These observations clearly show that light energy is helpful in nitrogen fixation and subsequent utilization of fixed nitrogen in the synthesis of aminoacids. Dhar and Mukerji (1935) and Dhar and Roy (1955) have obtained aminoacids by exposing to light the solutions of nitrate and glucose with titania as photosensitiser.

It is interesting to note that with the addition of phosphate to the system the number and amount of aminoacids is increased. This may be due to the fact that phosphate form more or less stable complexes with aminoacids and make these more stable (Dhar and Ghosh 1956) towards oxidation and decomposition. Dhar and Roy (1955) observed that in presence of phosphates, formation of amino-acids is enhanced. Das and Biswas (1957) found that the soil fertilized by phosphates was somewhat richer in aminoacids than the unmanured plots.

Similarly the systems inoculated with algae are also found to contain somewhat greater number and amount of aminoacids than the uninoculated ones. Serine was frequently detected in the algal sets containing organic matter and phosphates. Decomposition and oxidation of algal bodies may be responsible for the increased amount and number of aminoacids. Fowden (1951), Watanabe (1951), Williams and Burris (1952) and other workers found many free aminoacids present in algal bodies.

It is most fascinating to record that increase in nitrogen fixation due to inoculation of algae is very small as compared to the nitrogen fixed by the slow oxidation of organic matter aided by light and phosphates. Hence there is no doubt that the fertilization of the vergin soils and recouperation of nitrogen in tropical soils at least, is predominently a photochemical process.

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DESIGN OF ELECTRONIC SAMPLING SWITCH

By

D. C. SARKAR

Department of Physics, Muslim University, Aligarh

and

G. W. HUGHES

School of Electrical Engineering, Purdue University, Lafayette, Indiana, U. S. A.

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ABSTRACT

An electronic sampling switch with thirty-five input channels, each channel sampled for a period of 100 to 500 micro seconds, is designed as part of an analog to digital computer input system. The thirty-five stages of a ring counter are designed to provide gating voltage to diode gate circuits. The input signal voltages applied to the circuit vary from 10 mv. to 10 volts. The output voltage is found to be linear with respect to input voltage and the gain of the diode gate is found to be about 0·5.

INTRODUCTION

In connection with research on the recognition of speech by digital computer, we felt the need of designing an electronic sampling switch, whose mechanical analog is given in Fig. 1. In the figure it may be observed that there are thirty-five input channels and only one output. The rotating switch samples each channel for a period of say, 200 micro-seconds and within say, one or two micro-seconds the contact is made with the second channel. In this way the rotating switch will sample all thirty-five channels each for an equal interval of time and again repeat the operation. Mercury jet rotating switches are available on the market but are handicapped by a fixed and relatively low rate of sampling.

CIRCUIT

The block diagram of the circuit is given in Fig. 2. Negative pulses from a pulse generator trigger a thirty-five stage ring counter circuit. Gating waveforms from the plates of the ring counter circuit are applied to each of the thirty-five diode gating circuits. These thirty-five diode gate circuits have a common output.

PULSE GENERATOR

The pulse generator circuit consists of a free running multivibrator which produces square waves of variable repetition frequency as its plates. The square waves from one of the plates is differentiated by an R-C circuit, clipped by a diode clipper and amplified by a voltage amplifier to get a positive sharp pulse. This positive pulse is applied at the grid of 6L6 tube which is used as a power amplifier to deliver a negative sharp pulse of amplitude of 50 to 60 volts. This negative pulse is applied to the cathode of the left hand tube (*a*-section) of the ring counter. The sampling period is equal to the time between two consecutive triggering pulse which is determined by the time constant of the multivibrator. Therefore by changing this time constant, the sampling may be done at any desired rate. The pulse generator circuit is shown in Fig. 5.

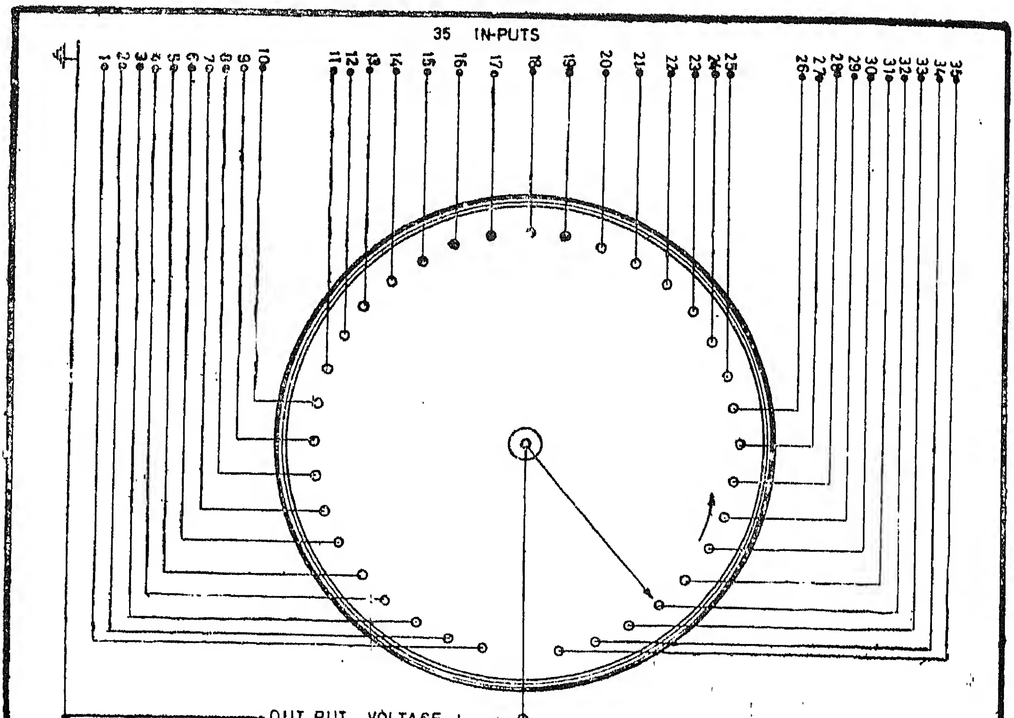


Fig No.1.

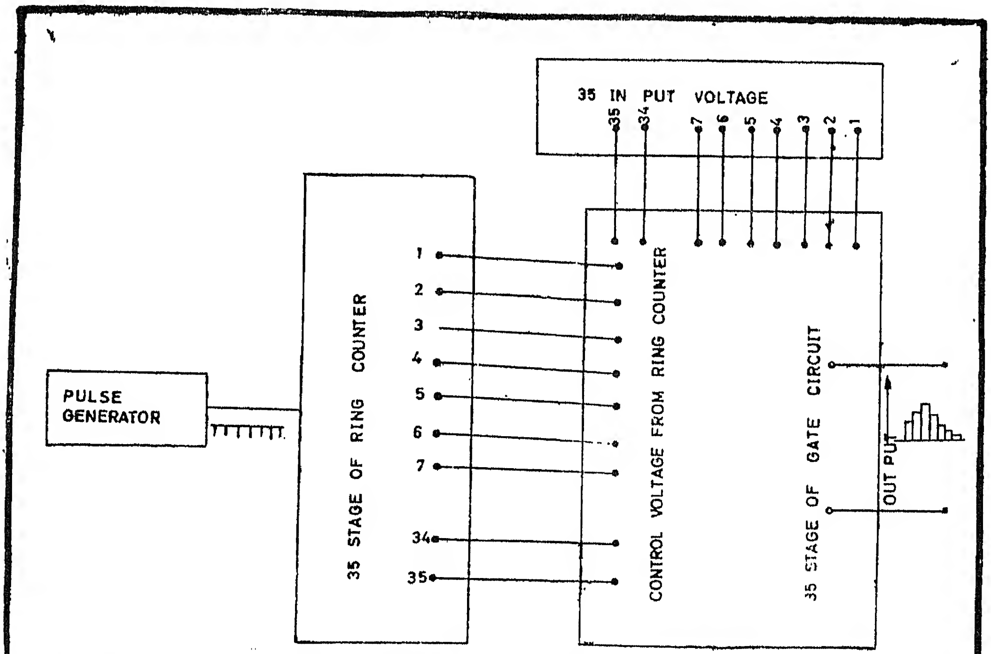


Fig No.2 BLOCK DIAGRAM OF ELECTRONIC SWITCH

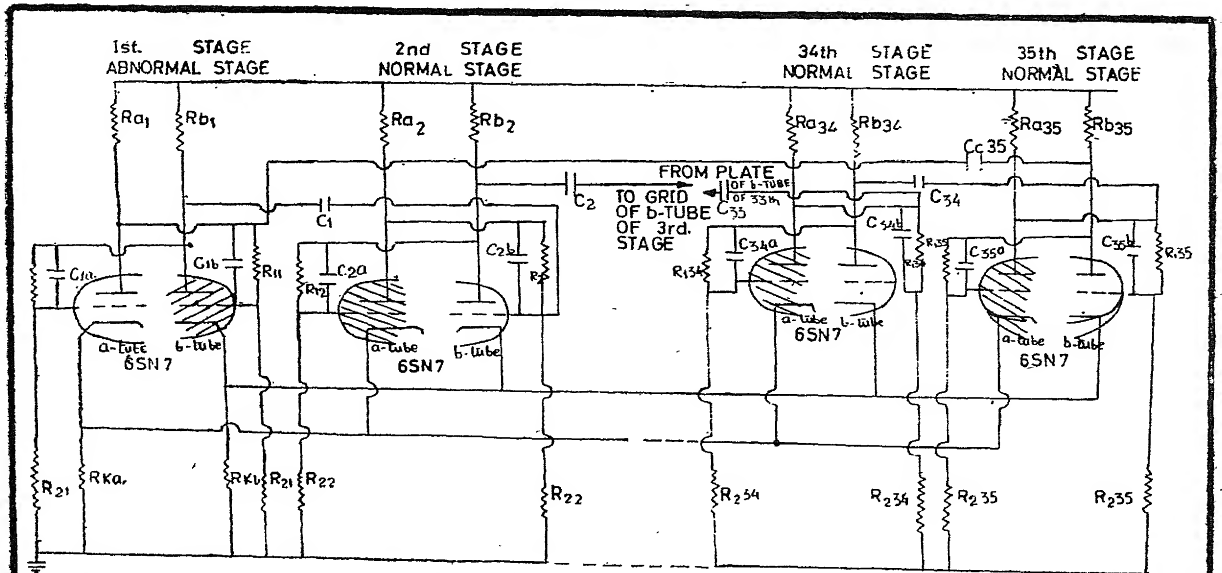
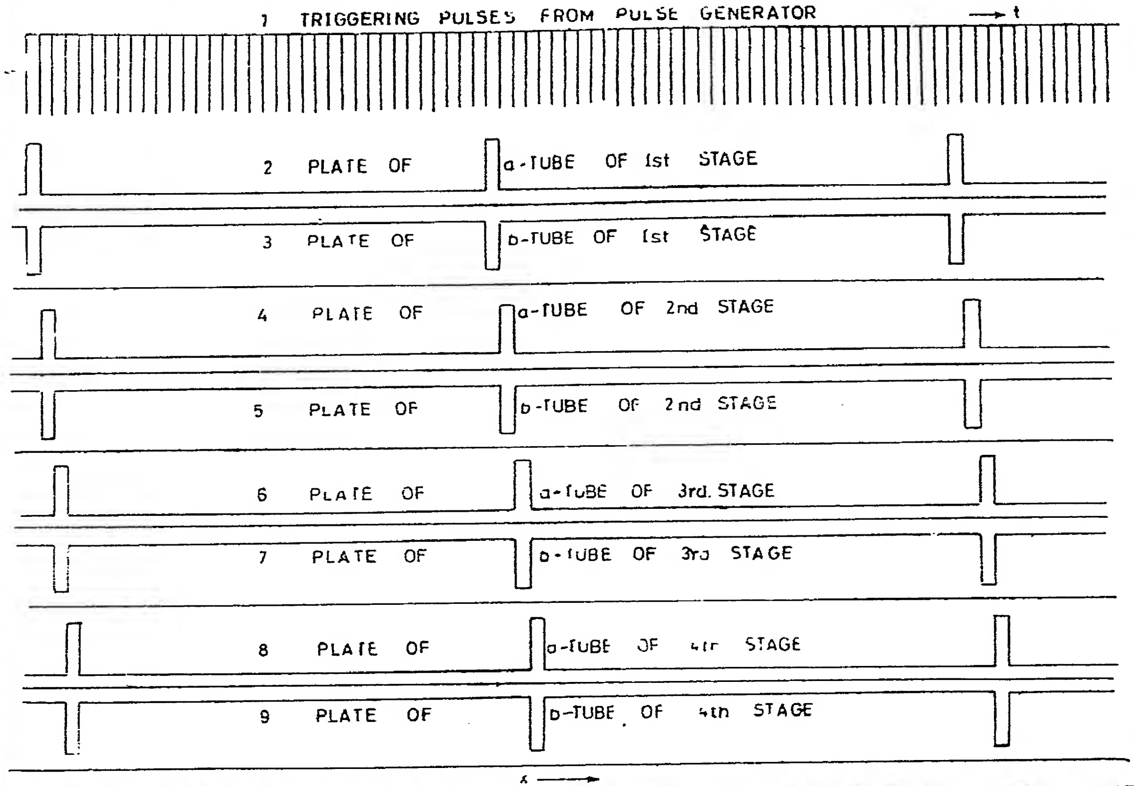


Fig No.3

RING COUNTER

Fig No - VOLTAGE WAVE FORMS



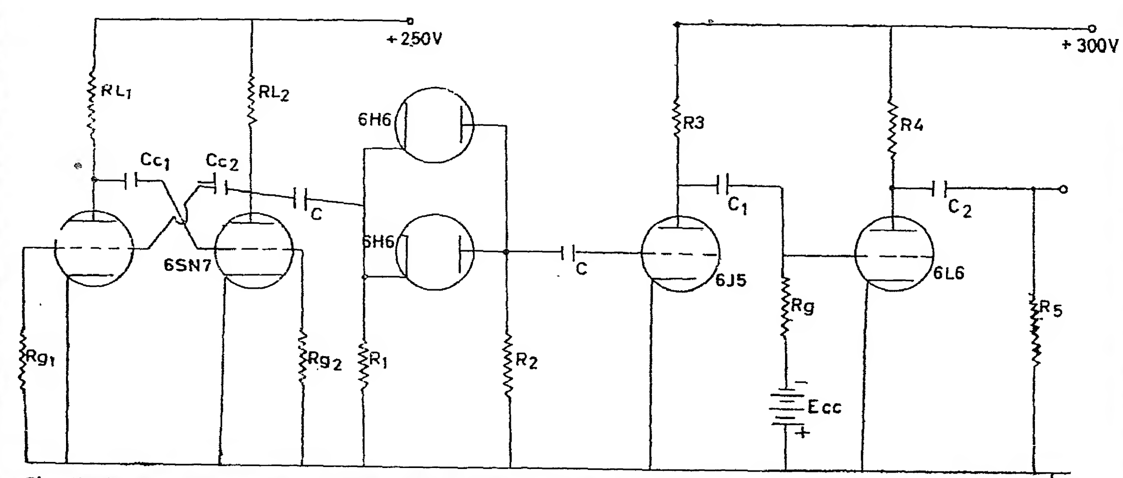


Fig No. 5 PULSE GENERATOR

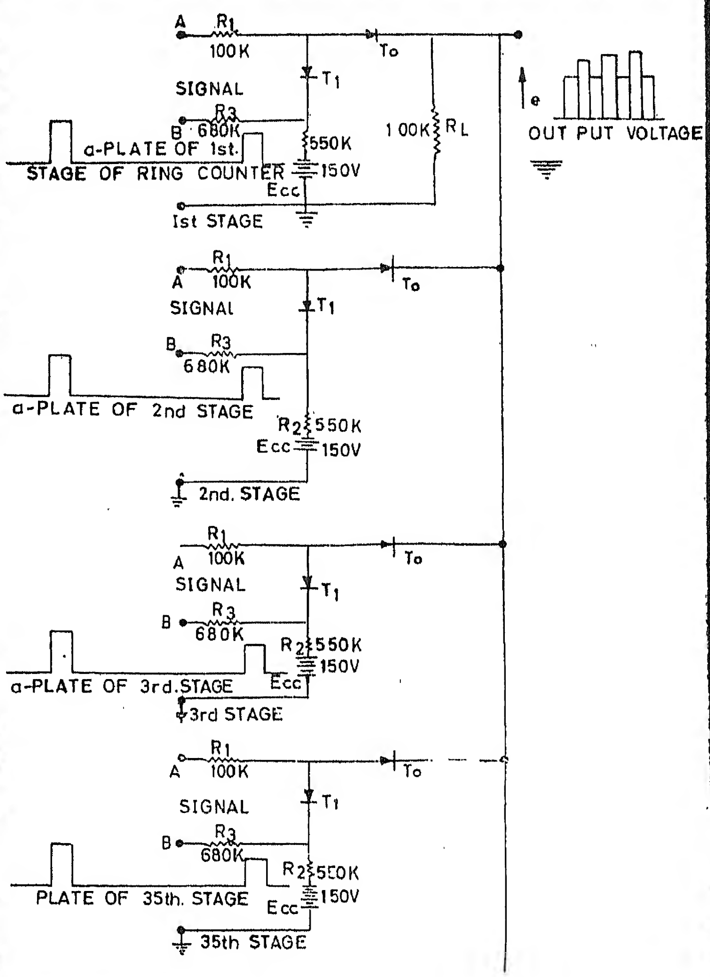


Fig. No.6 DIODE GATE CIRCUIT

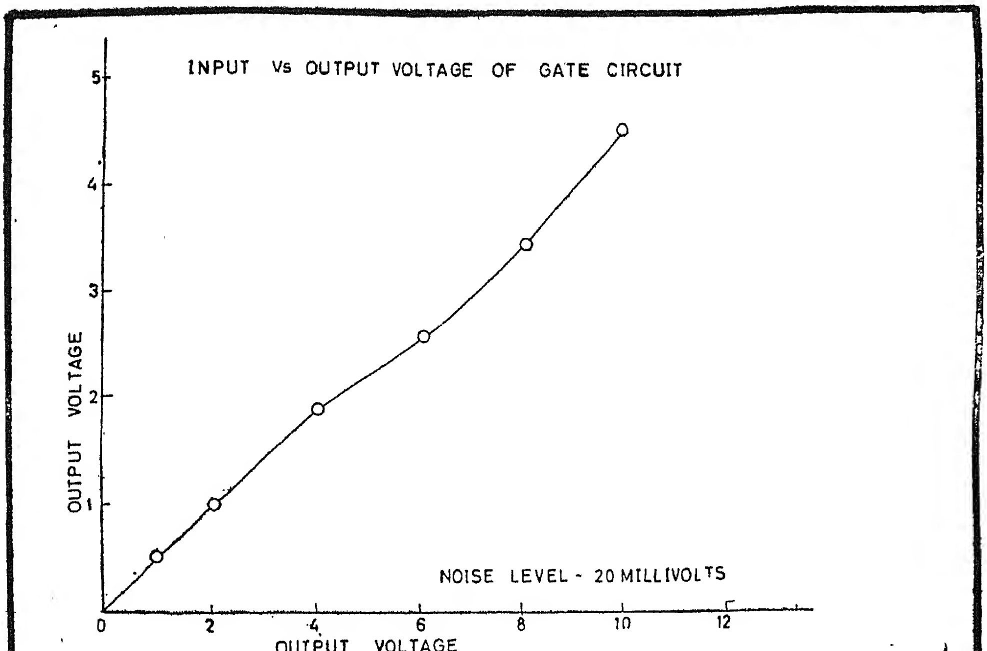


Fig No 7

RING COUNTER

In the ring counter there are thirty-five basic binary circuits arranged in the form of a ring with two common cathode resistances. For simplicity, the circuit of four stages of a ring counter and its wave-forms are shown in Fig. 3 and Fig. 4. It may be observed that the plate of the right hand tube of each binary is connected with the grid of the right hand tube of the next binary stage by means of a coupling capacitor and the plate of the right hand tube of the last stage is connected to the grid of the right hand tube of first stage completing the whole ring. Initially it is assumed that the right hand tube of the first binary stage (b–section) is conducting and the left hand tube (a–section) of the same stage is non-conducting ; while in every other binary stages, the right hand tube (b–section) is non-conducting, and the left hand tubes (a–section) of the same stages are conducting as shown in the figure by shading. If the current of a conducting tube in b–section is I_b the voltage drop across R_{kb} is $R_{kb}\ I_b$, since only one tube is sending current through R_{kb}. The total through R_{ka} from $(n-1)$ left hand tube (a–sectian) is $(n-1)\ I_a$ where n is the total number of the binary stages in the ring counter and I_a is the average current through each tube of a–section and the voltage drop across R_{ka} is $R_{ka}\ (n-1)\ I_a$. The second cathode resistance is so selected that $R_{kb}\ I_b = (n-1)\ R_{ka}\ I_a$. This is the most important point in the design of ring counter of n stages.

We considered that before application of the triggering pulse the circuit is operating with the shaded tube portion conducting such that the first stage is an abnormal one with a–tube off and b–tube on, while all other stages with a–tube on and b–tube off as shown in Fig. 3.

The application of the negative triggering pulse to the left hand cathodes (a–section) makes the second stage as an abnormal stage turning the first one as a normal stage. The triggering action of the ring counter is explained in detail in many textbooks. Thus the triggering pulse is bringing the second stage to the same condition as existed in the first stage before application of the triggering pulse. Similarly successive pulses applied to the cathode will produce similar type of stepping from one stage to the next down the line. The last stage is coupled to the first stage ; therefore the counter is brought back to the initial condition by the number of pulses equal to the number of stages and it will repeat the cycle. The waveforms at the plates of a–tube and b–tube of each stage are shown in Fig. 4.

Sometime difficulties are felt in triggering a ring counter of many stages. When the negative pulse is applied at the cathode, it turns on the a–tube of abnormal stage, turns off the b–tube of the same stage and also transfers current from the a-tube to the b-tube of the next stage. As the cathodes of all a–tube are connected together, this negative pulse will have the tendency to keep the a–tube of the second stage conducting whereas it is desired to change the a–tube to the non-conducting state. This does not occur due to the fact that the cross-over signal through the coupling capacitor is of such amplitude and duration as to over ride the effect of negative input pulse at the cathode. For this over riding action to take place it is necessary that the applied pulse shape may be maintained, and its amplitude can be controlled within quite close limits. The pulse shape and the amplitude control may be achieved by the pulse generator circuit of Fig. 5.

DESIGN AND ANALYSIS OF RING COUNTER

The gate voltage from the a-plates of the ring counter which is used as the control voltage of the diode gate circuit should have a value of about 100 volts. The values of different circuit constant are given below :

$Rb_1 = Rb_2 = Rb_3 = Rb_4 = \quad = Rb_{35} = 47$ kilohms.

$R_{a1} = R_{a2} = R_{a3} = R_{a4} = \quad = R_{a35} = 56$ kilohms.

$R_{11} = R_{12} = R_{13} = R_{14} = \quad = R_{135} = 470$ kilohms.

$R_{21} = R_{22} = R_{23} = R_{24} = \quad = R_{235} = 220$ kilohms.

$C_{1a} = C_{2a} = C_{3a} = C_{4a} = \quad = C_{35a} = 25$ micro-micro farads.

$C_1b = C_2b = C_3b = C_4b = \quad = C_{35}b = 25$ micro-micro farads.

$C_1 = C_2 = C_3 = C_4 = \quad = C_{35} = 75$ micro-micro farads.

$Rkb = 20$ kilohms.

A 67 kilohm load line is drawn on the characteristics of 6SN7 tube with the supply voltage of 200 volts. When the grid to cathode voltage of b-section tube is zero volt, the plate current Ib comes out to be 2·40 milliamperes. The plate voltage of b-section tube of abnormal stage of the ring counter is $eb_1 = 200 - 2{\cdot}40 \times 47 = 87$ volts and the voltage drop across the cathode resistance Rkb is $2{\cdot}40 \times 20 = 48{\cdot}0$ volts. The voltage across Rkb will have the tendency to make the grid of the b-section of the first stage negative with respect to cathode.

Let us check whether the b-section of the first stage is really at clamp.

The grid to earth voltage of b-section of first stage is equal to $\frac{200\times220}{470+220+56}=59$ volts. Before clamping the grid to cathode voltage is 59—48 = 11 volts. If we assume that the static grid to cathode resistance of the tube is 250 ohms, the grid to cathode voltage of the b-tube of the first stage after clamping is calculated to be

$$\frac{11\times0{\cdot}25}{{\cdot}25+\frac{220\times470}{220+470}}=0{\cdot}02 \text{ volts.}$$

So the tube is actually at clamp.

A 56 kilohm load-line is drawn on the characteristics of 6SN7 tube from the supply voltage of 200 volts. When $e_c=0$, the plate current is 2·9 milliampere. As the two sections of the cathode will remain at the same potential, we should have $2{\cdot}4\times20=34\times2{\cdot}9\times R_{ka}$ so that R_{ka} is approximately equal to 500 ohms. The voltage drop across R_{ka} is 49 volts and the cut-off value of the grid voltage corresponding to the plate-cathode voltage of (200—49)=151 volts is −9 volts. The grid to earth voltage of the a-section of first stage is $\frac{87\times220}{220+470}=27$ volts ; so the grid to cathode voltage of the same tube is $-49+27=-22$ volts, which is 13 volts below cutoff. Hence a negative pulse of such amplitude is essential for triggering that the grid of the cutoff tube will be raised up at least by 13 volts. When properly triggered the plate voltage waveform of all b-tube will vary from 87 volts to 200 volts, such that a pulse of 113 volts amplitude is available for the gate circuits.

Let us now check the operation of the normal binary stages, where a-tube is conducting and b-tube is non-conducting. From the 56 kilohm load-line the plate voltage of a-tube is $200-56\times2{\cdot}9=38$ volts. The voltage between the grid and ground of b-tube is $\frac{38\times220}{470+220}=12$ volts and the voltage across R_{kb} is 48 volts. Therefore, the grid to cathode voltage of the b-tube is $12-48=-36$ volts, which is far below cutoff. Let us see whether a-tube of the normal stages is really at clamp. The voltage across the grid to ground of a-tube is $\frac{220\times200}{47+470+220}=60$ volts. Before clamping its grid to cathode voltage is $60-48=12$ volts. If we asssume that the static grid cathode resistance is 250 ohms, the grid cathode voltage of b-tube after clamping is calculated to be

$$\frac{12\times0{\cdot}25}{0{\cdot}25+\frac{220\times470}{220+470}}=0{\cdot}02 \text{ volts.}$$

When proper triggering voltage is applied at the cathode of a-tubes, the plate voltage wave-forms of all a-tube will vary from 38 volts to 200 volts such that a gate voltage of 162 amplitude is available for the diode gate circuits.

DIODE GATE CIRCUIT

The circuit diagram of the gate circuit is shown in Fig. 6. As in this case the input voltage is a d-c voltage from 20 millivolts to 10 volts, a unidirectional gate circuit is used. The above circuit works in the following way : As observed in the figure the input voltage is applied at the point A and a positive gate signal from the a-plate of the ring counter is applied at the control terminal B. Initially when the voltage at the control terminal B is very low, the diode T_1

conducts, since the voltage at the cathode of T_1 is adjusted to a negative value by the negative biasing voltages E_{cc} which is about – 150 volts and the resulting current through R_1 back biases the diode T_0. Therefore, in absence of a gate pulse, no input voltage will be transmitted to the output, provided the input voltage does not exceed certain value. A positive going gate signal, which is chosen to be of such amplitude as to back bias T_1, causes the diode T_1 to cease conduction and the gate, therefore, is open for transmission. The input signal must be *d·c* coupled, but the gate may be either *a·c* coupled or *d·c* coupled. But in this case we preferred a *d·c* coupled one. In this way thirty-five stages of identical gate circuits are constructed such that all stages have common load resistance RL. The control signal or the gate voltages to each gate circuit is provided from the plates of each left hand side tube (*a*-section) of the thirty-five binaries of the ring counter. We know that as the successive pulses are applied to the cathode of the ring counter, the gate is transferred from one stage to the next and when the gate has shifted to the last stage, the cycle is repeated again. As the plate of the left hand tube of first stage is connected to the first stage of the diode gate circuit, during this gate period the gate circuit no. 1 will remain open for transmission, and all other circuits are closed during that time. During the second gate period, the gate circuit no. 2 will remain open for transmission and all other circuits except no. 2 will remain closed for transmission. The gain of the gate circuit is $R_1\ (R\text{L}+R\text{f}+R_1)$ where Rf is the forward resistance of the diode T_0.

EXPERIMENTAL RESULTS

The above circuit is constructed. The most important parameters in the operation of the ring counter are the cathode resistances R_{kb} and R_{ka}. In the actual circuit R_{ka} and R_{kb} are taken as the variable resistances such that their values may be varied within 20% of their calculated values. By adjusting the cathode resistances, supply voltages and triggering pulse height, proper gating voltage waveforms may be observed at the *a*-plates of the ring counter. The resistances chosen in the gate circuits have a large value such that they do not load the ring counter. From a voltage divider some input voltage applied at the input terminal of the gate circuits. The output voltage is observed on a oscilloscope. The curve between the input and the output voltage is shown in Fig. 7. The noise level if found to be 20 millivolts and the gain of the diode gate circuit is found to be about 0·5.

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ON THE CONSTITUTION OF METAL HYDROUS OXIDE SOLS AS DETERMINED BY ELECTROMETRIC TITRATIONS

By

WAHID U. MALIK, S. K. SRIVASTAVA *and* G. C. GUPTA

Chemical Laboratories, University of Roorkee, Roorkee

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ABSTRACT

Potentiometric titrations of alumina and ferric oxide sols have been carried out seperately as well as with the mixtures of the two sols in aqueous and non aqueous medium using both the glass and hydrogen electrodes. From the number of inflexion points it is concluded that the alumina sol behaves like a dibasic acid in the alcoholic medium but no change from monobasic to dibasic character is observed with the ferric oxide sol. The results of potentiometry also find confirmation with the conductiometric data on the titrations of the two sols. With alumina ferric oxide sol mixtures two inflexion points are realised with mixtures containing upto 20% alumina. Below this ratio only one inflexion point is obtained.

Data on Pallmann Effect studied with different concentrations of the sols does not indicate this difference in behaviour as far as measurements in two medium are concerned.

The constitution of metal hydroxide sols had been a subject of much controversy. A number of workers, namely Wyrouhoff and Verneuil[1], Duclaux[2], Malfitano[3], Hantzsch and Desch[4], Linder and Picton[5] had suggested that these sols contain basic salts or chlorides of condensed hydroxides. Pauli[6] on the other hand was of the view that the colloidal particles in hydroxide sols could be compared to complex ions resulting from the ionisation of substances closely related to coordination compounds. The most significant contribution in this direction had been of Thomas and Coworkers[7,8] who in a series of communications had tried to explain the phenomenon of the formation (including peptization), precipitation and dissociation of hydrous metal oxides in terms of olation, oxolation and anion penetration. Whitehead[9] had made a comparative study of the complex compound theory of hydrous oxides with other theories on the formation and precipitation of colloidal solutions.

On the experimental side, most of the work of Thomas, cited above deals with pH measurements of metal hydroxide sols prepared under different conditions and subjected to such treatment as ageing, boiling, dilution, peptization by different electrolytes, coagulation by strong and weak electrolytes and charge reversal etc. They have however, done little to elucidate their view point by the electrometric titration technique. It was, therefore, thought desirable to resolve this problem with the help of pH metric titrations carried out in the aqueous and alcoholic media. The results obtained with these studies have been confirmed by the conductometric titration method. Attempt is also made to study the phenomenon of Pallmann effect vis-a-vis the composition of sols. These studies, however, failed to give any useful information on this point.

EXPERIMENTAL

1. *Alumina sol.*—Ten gms. of anhydrous aluminium chloride (E. Merck) were dissolved in about 200 ml. of double distilled water to which dilute ammonium hydroxide was added with vigorous stirring and simultaneous boiling.

The white gelatinous ppt. of $Al(OH)_3$ thus obtained was centrifuged and washed free of the dissolved salts. After thorough washing the ppt. was suspended in about 500 ml of double distilled water, boiled and to it were added equal quantities of 1N HCl and H_2O, till the ppt. was completely peptized. It was kept boiling throughout peptization. Afterwards it was dialysed and the pH maintained at 1·70 after dilution of this to one litre.

2. *Ferric sol.*—About 800 ml of double distilled water was boiled in a Pyrex glass beaker and to this was added $FeCl_3$ 50% solution drop-wise with continuous boiling and stirring so as to allow sufficient time for its hydrolysis. In every case 2 gm. of $FeCl_3$ was added. Next the sol on cooling was dialyzed diluted to one litre and the pH maintained at 2·50.

3. *Alkali solutions.*—In every case N/5 KOH in double distilled water or double distilled ethyl alcohol was used as the titrant.

Titrations were done in about fifteen pyrex boiling tubes. Ten ml. of the pure or the mixed sols in the given ratios (9 : 1, 8 : 2, 2 : 8, 1 : 9, etc.) were added to the tubes. To each tube were added different amounts of the titrant and the volume was made up to 20 ml. in each case either with water or alcohol. The tubes were well corked and in the first instance readings were taken for pH measurements with the Cambridge Bench type pH meter with Cambridge glass calomel electrodes and also by the Hydrogen electrode. Conductivity was measured with a Kohlarausch Bridge type Philips conductivity meter. In case of Pallmann effect studies the solutions were made in 25 ml. Pyrex beakers which were covered and kept aside for six hours at a lower temperature. The sol always coagulated due to the action of alkali and in no case kept floating on the surface as the medium used was partly aqueous. The pH of the supernatant equilibrium liquid and that of the settled ppt. or coagulam was measured with the glass calomel electrode assembly, once keeping the assembly in contact with the liquid and then with the suspension.

The values of Base-Exchange-Capacities have been obtained from the inflexion points in the curves and those of dissociation constants have been calculated from the relation.

$pH = pK + \log \frac{\text{salt}}{\text{acid}}$ so that at half neutralization of the first acidity $pH = pK$ and so on. No difference was found in the results obtained with the fresh sol and the aged sols (sols kept for 48 hours). Results are given in the table I

DISCUSSION

Typical pH titration curves are obtained on titrating ferric oxide and alumina sols against caustic potash. (Fig. 1. curves 1–4). There is however a striking difference in the alumina curves realized in the aqueous and non-aqueous media. It will be seen that the number of inflexion points are increased from one to two in passing from the aqueous to the alcoholic medium. Although Ruehle[10], Fritz,[11] Palit[12] and Singhal and Malik[13] have recommended non aqueous titrations in certain cases like polybasic acids and clay minerals in order to avoid the levelling effect of the aqueous medium, there appears to be little justification to explain the results of the titrations of hydrous oxide sols on this basis. Even if it is assumed that these sols can behave as dibasic acids, the very fact that the pK values for the alumina sol (as determined from the two inflexion points) are stretched so apart (Table I) that the levelling effect of water can

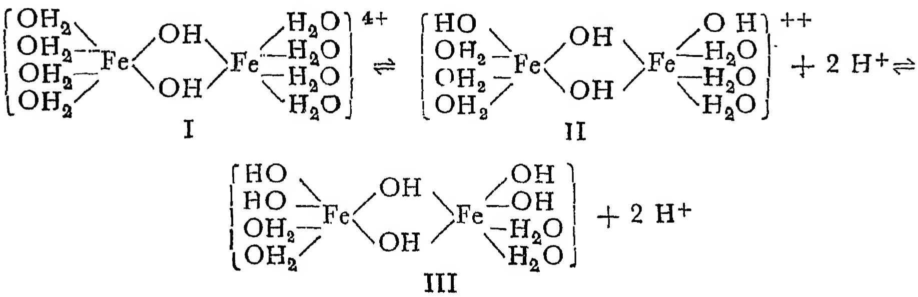
hardly interfere with the emergence of the two inflexion points. The reason for the difference in behaviour in the two media should, therefore, lie somewhere else.. Assuming that these sols are metallic oxy chloride sols rather than hydrous oxides (in other words the chloride ions are necessary for their stability) their formation may be represented as follows :

Ferric oxide sol.

1. $Fe(H_2O)_6^{+++} \rightleftharpoons Fe(H_2O)_5(OH)^{++} + H^+$

2. $Fe(H_2O)_5(OH)^{++} \times 2 \rightleftharpoons (OH_2)_4Fe\langle{}^{OH}_{OH}\rangle Fe(H_2O)_4^{++++} + 2H_2O$

Reaction (2) may be considered as one responsible for the formation of micelles in this sol. The sol complex thus formed undergoes the following changes due to further hydrolysis to give the colloidal solution.



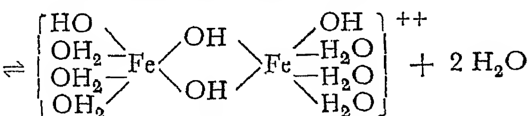
I II III

Since H^+ act as peptizing ions in the formation of the sol, an equilibrium represented by I and II above should exist. The ol polymer shown by III would represent the coagulated state of the sol, that is, one from which all the peptizing ions have been removed resulting in the formation of an uncharged micelle. Existence of one inflexion point in the pH titration curves support the above mechanism.

The other possibilities *viz.* the formation of polymeric ol or oxo compounds is ruled out since these would involve the existence of more than one inflexion point in the pH titration curves.

3. $Fe(H_2O)_5OH^{++} \rightleftharpoons Fe(H_2O)_4(OH)_2^+ + H^+$

4. $2\,Fe(H_2O)_5(OH)_2^+ \rightleftharpoons$

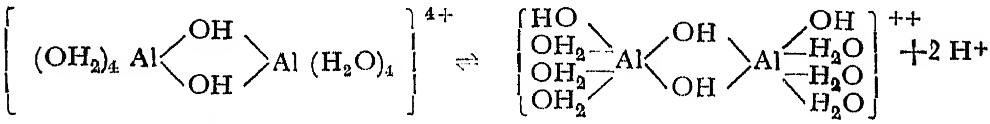


IV

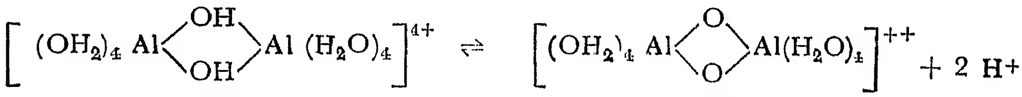
The complex ol compound (IV) undergoes further hydrolysis which may give a complex corresponding to III representing again a coagulated state of the sol.

Although the possibility of the existance of oxo compounds does not exist in the case of ferric oxide sols (since only one inflexion point appears both in the aqueous and non aqueous media) it is quite possible in the case of the alumina sol. Assuming that just as in the case of ferric oxide sol here colloidal formation

takes place according to reactions (1) and (2) and not according to (3) and (4), the only possible cause for having two inflexions in the titration curves of alumina in the alcoholic medium is the existence of oxo compounds together with the ol compounds in the alumina sol.



IV V



VI VII

The existence of the equilibria IV and V and also VI and VII may be considered responsible for the two inflexion points in the alcoholic titrations. The fact that the process of oxolation which involves the loss of proton from the ol bridge is facillitated by the addition of alcohol as shown by Hall and Eyring[14] also lends support to the mechaism proposed above. Moreover the tendency of the ol complex to change into the oxo complex in the non aqueous medium is so marked that this behaviour is exhibited even in mixtures of alumina and iron oxide containing as low a proportion of alumina as 2 alumina : 8 ferric oxide. (Table 1).

TABLE I

Base Exchange Capacity[1] and pK[2] values calculated from pH metric and conductimetric curves.

1. m.e.q. per litre of the sol.

2. pK values at the respective inflexion points are given in parentheses.

Sol	Aqueous		Alcoholic	
	pH	Conductivity	pH	Conductivity
Alumina	600 (2·2)	605	600 (3·1); 1190 (7·4)	600 ; 1200
Ferric	500 (3·1)	500	510 (3·7)	500
9 Al : 1 Fe	595 (2·1)	590	590 (2·5); 1180 (7·7)	600 ; 1180
8 Al : 2 Fe	580 (2·3)	580	585 (2·8); 1170 (8·0)	580; 1160
7 Al : 3 Fe	575 (2·4)	570	570 (2·9); 1150 (8 5)	575 ; 1150
6 Al : 4 Fe	560 (2·5)	560	565 (2·8); 1130 (8·2)	560 ; 1130
5 Al : 5 Fe	555 (2·5)	550	550 (2·9); 1110 (1·4)	550 ; 1100
4 Al : 6 Fe	545 (2·7)	540	550 (3·0); 1080 (7·5)	540 ; 1090
3 Al : 7 Fe	530 (2·7)	530	530 (3·1); 1050 (7·7)	530 ; 1070
2 Al : 8 Fe	520 (2·6)	520	525 (3·2); 1040 (7·8)	500 ; 1040
1 Al : 9 Fe	515 (2·9)	510	510 (3·2)	510

pH metric
Titration Curves

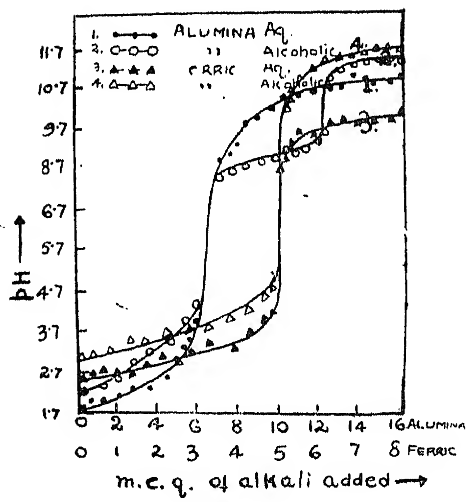


Fig. 1

Conductimetric
Titration Curves

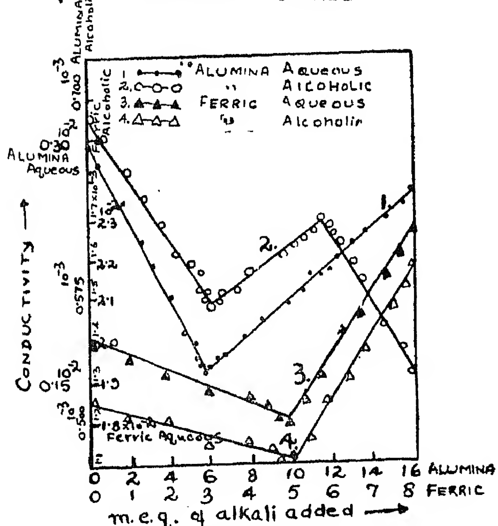


Fig. 2.

Pallmann Effect

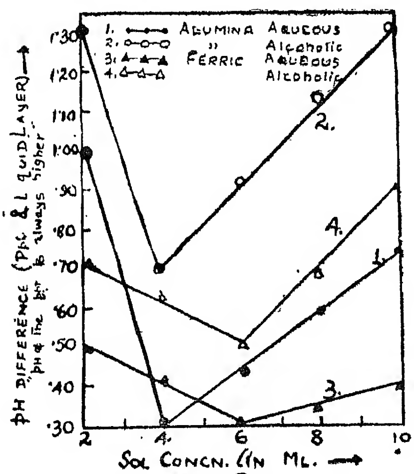


Fig. 3

Further confirmation to the view point discussed above is forthcoming from the results on the conductometric data in aqueous and alcoholic media (Fig. 2 curves 1–4). Here too the behaviour of alumina sol as such and in its mixtures with ferric oxide is similar to that of the pH metric titrations.

The inflexion in the pH metric and conductometric titrations of clays with alkalies is often taken as a measure of the base exchange capacity of clays. If this can be taken as a measure of such determination then the exchange values (for the exchange of chloride ion by hydroxide ion of the alkali) given in table 1., may have some significance. However, the similarity of the data obtained from pH-metric and conductometric titrations is particularly striking. The data on pK values on the assumption that the hydrous oxide sols of alumina and ferric oxide are the dissociation products of the ol and oxo compounds are also enumerated in table 1.

The studies on Pallmann effect do not provide any new information. The nature of the curves between pH and concentration of sol is the same both for aqueous and non aqueous media. (Fig. 3 curves 1–4).

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EFFECT OF ORGANIC SUBSTANCES ON NITRIFICATION BY NITROSOMONAS. PART—III

By

S. P. TANDON *and* R. C. RASTOGI

Department of Chemistry, University of Allahabad, Allahabad

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ABSTRACT

The effect of glucose. fructose, mannose, galactose, xylose, arabinose, lactose, maltose, mannitol and tartaric acid on the growth and activity of *Nitrosomonas* has been studied. It has been observed that smaller concentrations of these organic substances produce beneficial effects on the growth and activity of *Nitrosomonas*. The process of nitrite-formation is hindered in the beginning for a certain period but after that it begins and takes place even more vigorously than when no sugar is added. The results show that the bacteria feed upon sugars and other organic material first and it is only when these have been used up that they begin to derive their energy for their growth and metabolism from the oxidation of ammonium salts. Even in such a high concentration of sugars as 1000 mg/82 ml the bacteria do not get killed, but only remain dormant.

In previous two communications[1,2] relating to the "Effect of orgainc substance on *Nitrosomonas*, Part I and II", the results showing the effect of D-glucose, D-fructose, D-mannose, D-galactose, L-xylose and L-arabinose on the above bacteria were given. In the present paper the results of our study of nitrite formation by *Nitrosomonas* in presence of lactose, maltose, mannitol and tartaric acid are recorded.

The experimental procedure adopted was the same as described in Part I.

TABLE 7A

Nitrosificition in the presence of lactose

(*i*) Volume of the culture medium taken = 80 ml
(*ii*) Volume of the enriched culture (inoculum) added = 1 ml
(*iii*) Volume of the ammonium sulphate added = 1 ml

No.	Amount of lactose added to the medium (in mg)	Amount of lactose left at different intervals of time Time in hours							
		48	96	144	192	240	288	336	384
1	Control	Nil	Nil	Nil	Nil	Nil	Nil	Nil	Nil
2	5·00	,,	,,	,,	,,	,,	,,	,,	,,
3	10·00	6·785	,,	,,	,,	,,	,,	,,	,,
4	20·00	14·93	10·85	5·428	,,	,,	,,	,,	,,
5	25·00	21·71	17·64	10·85	8·142	,,	,,	,,	,,
6	30·00	27·14	23·06	19·00	10·85	6·785	,,	,,	,,
7	40·00	35·28	29·85	22·21	14·62	6·785	,,	,,	,,
8	50·00	43·424	36·64	31·21	19·00	12·21	5·428	,,	,,
9	60·00	52·92	44·78	37·996	31·211	27·14	16·28	6·785	,,
10	80·00	77·35	59·71	52·92	48·85	39·353	31·21	19·00	10·85
11	100·00	96·34	88·26	82·777	67·85	59·70	51·56	43·42	29·85

Control = Containing no lactose

TABLE 7B

Nitrosification in the presence of lactose

(*i*) Volume of the culture medium taken = 80 ml
(*ii*) Volume of the enriched culture (inoculum) added = 1 ml
(*iii*) Volume of the ammonium sulphate added = 1 ml

No.	Amount of lactose added to the medium (in mg)	Nitrite formed at different intervals of time (mg/litre) Time in hours 48	96	144	192	240	288	336
1	Control	0·92	1·8400	2·3096	6·6164	24·334	50·324	138·00
2	5·00	–	0·782	1·4375	4·600	24·642	62·790	209·461
3	10·00	–	–	0·92	1·7972	9·5818	45·176	123·28
4	20·00	–	–	–	0·782	2·2117	9·5818	39·56
5	25·00	–	–	–	–	1·1696	3·220	24 336
6	30·00	–	–	–	–	1·0120	2·300	17·572
7	40·00	–	–	–	–	–	1·4825	6·900
8	50·00	–	–	–	–	–	1·012	1·916
9	60·00	–	–	–	–	–	–	1·0120
10	80·00	–	–	–	–	–	–	–
11	100·00	–	–	–	–	–	–	–

Control = Containing no lactose

TABLE 8A

Nitrosification in the presense of maltose

(*i*) Volume of the culture medium taken = 80 ml
(*ii*) Volume of the enriched culture (inoculum) added = 1 ml
(*iii*) Volume of the ammonium sulphate added = 1 ml

No.	Amount of maltose added to the medium (in mg)	Amount of maltose left at different intervals of time Time in hours 48	96	144	192	240	288	336	384
1	Control	Nil	Nil	Nil	Nil	Nil	Nil	Nil	Nil
2	5·00	,,	,,	,,	,,	,,	,,	,,	,,
3	10·00	6·785	,,	,,	,,	,,	,,	,,	,,
4	20·00	14·927	5·428	,,	,,	,,	,,	,,	,,
5	25·00	20·125	12·21	,,	,,	,,	,,	,,	,,
6	30·00	23·069	17·64	12·21	5·428	,,	,,	,,	,,
7	40·00	35·28	28·50	19·00	13·57	6·785	,,	,,	,,
8	50·90	44·78	39·353	32·568	21·71	11·36	5·428	,,	,,
9	60·00	52·92	47·49	42·06	37·99	29·85	20·350	13·57	6·785
10	80·00	71·92	67·85	59·71	52·92	43·424	33·925	27·14	14·92
11	100 00	96·38	88·205	76·00	67·85	59·71	52·92	44·78	31·211

Control = Containing no maltose

TABLE 8B

Nitrosification in the presence of maltose

(*i*) Volume of the culture medium taken = 80 ml
(*ii*) Volume of the enriched culture (inoculum) added = 1 ml
(*iii*) Volume of the ammonium sulphate added = 1 ml

No.	Amount of maltose added to the medium (in mg)	Amount formed at different intervals of time (mg/litre) Time in hours 48	96	144	192	240	288	336
1	Control	0·920	4·1035	19·163	36·80	73·862	103·86	172·56
2	5 00	–	1·7967	7·682	35·139	86·85	146·625	211·60
3	10·00	–	0·920	1·964	14 375	49·831	95·827	153·52
4	20·00	–	–	1·104	7·682	17·9676	28·75	55·52
5	25·00	–	–	–	0·92	3·4120	14·375	41·028
6	30·00	–	–	–	–	1·0551	6·7055	17·228
7	40·00	–	–	–	–	–	2·30	7·682
8	50·00	–	–	–	–	–	–	1·1040
9	60·00	–	–	–	–	–	–	–
10	80·00	–	–	–	–	–	–	–
11	100 00	–	–	–	–	–	–	–

Control = Containing no maltose

TABLE 9A

Nitrosification in the presence of D-mannitol

(*i*) Volume of the culture medium taken = 80 ml
(*ii*) Volume of the enriched culture (inoculum) added = 1 ml
(*iii*) Volume of the ammonium sulphate added = 1 ml

No.	Amount of D-mannitol added to the medium (in mg)	Amount of mannitol left at different intervals of time mg/litre Time in hours 48	96	144	192	240	288	336	384
1	Control	Nil	Nil	Nil	Nil	Nil	Nil	Nil	Nil
2	5·00	,,	,,	,,	,,	,,	,,	,,	,,
3	10·00	6·038	,,	,,	,,	,,	,,	,,	,,
4	20·00	13·796	5·890	,,	,,	,,	,,	,,	,,
5	25 00	19·146	8·10	,,	,,	,,	,,	,,	,,
6	30·00	25·03	17·672	8·542	,,	,,	,,	,,	,,
7	40·00	36·819	29·46	17·672	6·036	,,	,,	,,	,,
8	50·00	46 655	36 819	27·982	17·672	5·890	,,	,,	,,
9	60·00	57·436	51·546	41·237	30·04	19·146	4·712	,,	,,
10	80·00	79·52	73·636	70·691	64·80	57·436	48·60	39·765	27·98
11	100·00	100·14	97·20	92·783	86·891	81·00	73·636	64·80	53·019

Control = Containing no mannitol

TABLE 9B

Nitrosification in the presence of D-mannitol

(*i*) Volume of the culture medium taken	=	80 ml
(*ii*) Volume of the enriched culture (inoculum) added	=	1 ml
(*iii*) Volume of the ammonium sulphate added	=	1 ml

No.	Amount of D-mannitol added to the medium (in mg)	Nitrite formed at different intervals of time (mg/litre) Time in hours						
		48	96	144	192	240	288	336
1	Control	0·920	1·335	6·729	21·184	39·46	79 25	189 36
2	5·00	1·575	1·150	5·966	19·230	43·208	90·99	230·00
3	10·00	–	0·7187	2·300	10·181	34·731	81·36	211·60
4	20·00	–	–	1·150	3·274	14·375	54·167	115·16
5	25·00	–	–	0·6388	1·4375	9·200	35·41	92·91
6	30·00	–	–	–	0·575	3·334	11·50	42·60
7	40·00	–	–	–	–	0·760	2·750	16·67
8	50·00	–	–	–	–	–	0·920	4·7916
9	60·00	–	–	–	–	–	–	1·000
10	80·00	–	–	–	–	–	–	–
11	100·00	–	–	–	–	–	–	–

Control = Containing no mannitol

TABLE 10

Nitrosification in the presence of D-tartaric acid

(*i*) Volume of the culture medinm taken	=	80 ml
(*ii*) Volume of the enriched culture (inoculum) added	=	1 ml
(*iii*) Volume of the ammonium sulphate added	=	1 ml

No.	Amount of D tartaric acid added to the medium (in mg)	Nitrite formed at different intervals of time (mg/litre) Time in hours						
		48	96	144	192	240	288	336
1	Control	1·9167	4·423	12·321	33·541	79·86	184·[illegible]0	345·00
2	5·00	1·600	5·75	18·160	60·375	106·45	230·00	419·00
3	10·00	1·4375	2·13	5 227	21·81	57·50	95·934	239·58
4	20·00	1·240	1·5131	2·875	8·214	26·54	65·71	140·00
5	25·00	1·1051	1 337	1·800	4·107	19 17	46·00	102·33
6	30·00	0·989	1·150	1·474	3·194	8·214	23·00	52·272
7	40·00	0·828	1·00	1·278	2·211	4·192	12·78	31·94
8	50·00	0·690	0·943	1·15	1·800	3·0267	8·214	22·115
9	60·00	0·690	0·943	1·085	1·643	3·0267	7·19	19·167
10	80·00	0·46	0·736	0·989	1·15	2·30	4·107	11·50
11	100·00	0·46	0·69	0·989	1·085	1·4375	1·9167	4·423

Control = Containing no tartaric acid

TABLE 11

Nitrosification in the presence of large concentration of D-glucose

(*i*) Volume of the culture medium taken = 80 ml
(*ii*) Volume of the enriched culture (inoculum) added = 1 ml
(*iii*) Volume of the ammonium sulphate added = 1 ml

Concentration of D-glucose

No.	Amount of D-glucose added to the medium (in mg)	Amount of D-glucose left after 840 hours (in mg)	Nitrite formed in 840 hours mg/litre
1	Control	–	848·40
2	100·00	–	836·90
3	200·00	–	810·00
4	300·00	58·00	–
5	400·00	212·00	–
6	500·00	367·00	–
7	600·00	516·00	–
8	800·00	742·00	–
9	1000·00	962·00	–

Control = Containing no glucose

DISCUSSION

Warington[3] was the first to report that the presence of sugar produced an injurious effect upon the process of nitrification. It was later supported by Winogradsky[4] who observed that 0·015 M of glucose completely stopped the nitrification by *Nitrosomonas* in pure culture and the solution became sterile. Among other workers who produced evidence in support of the injurious effect of organic compounds in general upon nitrification are Pandalai, Nelson, Ruban and Gundersen. Pandala[5] found that nitrification was depressed in the presence of organic matter. Nelson[6] found that 1% dextrose and 0·5% pentoses completely stopped the bacterial nitrification and the culture became sterile. He further observed that lower concentrations of these sugars only retarded the initial growth of the bacteria but did not produce any permanent toxic effect. Gundersen[7] has produced results showing that although certain organic compounds such as tryptophan, glutamic acid, tyrosine, phenylalanine and histidine cause inhibition of nitrite formation when they are present in the culture, yet certain other organic compounds like vitamins, such as thiamine, riboflavine, p-aminobenzoic acid, nicotinamide, pyridoxine, pantothenic acid and vitamin B_{12} do not produce any inhibitory effect on the process of nitrification by *Nitrosomouas*. Ruban[8] reported that although in his experiments the presence of organic compounds stopped the oxidation of ammonium salts by nitrite formers, yet the organism did not get killed and when transferred to a fresh culture medium they resumed their growth and activity.

Sugars and various other organic compounds serve as universal food material for living organisms. It is, therefore, difficult to understand why organic food material should prove toxic or inhibitory to the growth of *Nitrosomonas*. A detailed study is, therfore, needed to find out how far the organic food material is really toxic to the nitrite formers.

Like every living organism *Nitrosomonas* also needs energy for its growth and metabolism. The physiological significance of the oxidation of ammonium sulphate to nitrite by *Nitrosomonas* is merely to furnish energy to bacteria. Organic food materials required by other organisms also serve the same purpose. If organic food materials are assumed to be injurious to *Nitrosomonas*, it means they do not serve as food for these bacteria, while ammonium salts serve as food for them.

In order to understand the influence of organic food material on nitrite formers, we have carried out extensive experiments with *Nitrosomonas* in the presence of varied amounts of glucose, fructose, mannose, galactose, xylose, arabinose, lactose, maltose mannitol and tartaric acid.

The results obtained by us are recorded in the Tables 1A, 1B, 2A and 2B, (Part I), Tables 3A to 6B (Part II) and Tables 7A to 11 (Part III). They do not substantiate the view that the organic food materials are toxic to the growth of the nitrite formers. On the other hand they are found to be beneficial when present in small concentrations (from 5 to 100 mg/82 ml).

Thus our results show that when the sugars are present in small concentration they hinder the process of nitrite formation in the beginning for a certain period but after that period the nitrite formation begins and takes place even more vigorously than when no sugar is added. We are led to conclude that the bacteria utilize the sugars in the beginning as food and that after the sugars have been consumed they begin to derive energy for their growth and metabolism from the oxidation of ammonium sulphate to nitrite. These results clearly indicate that sugars in proper amounts serve as better food material for *Nitrosomonas* than does ammonium sulphate. The bacteria feed upon the sugars first as a result of which the latter are oxidised and thus energy is liberated for the organism. It is only when these sugars have been consumed that the bacteria begin to oxidise ammonium salts (here ammonium sulphate) for obtaining their energy. That sugars are better food for these bacteria than the ammonium sulphate is clear from the results contained in the Tables 1 to 10, which show that the formation of nitrite takes place more vigorously in the samples which contain sugars (5 mg to 10 mg) than in those which do not contain sugars after a period when the sugars in the former samples have been consumed. The higher concentration of nitrite found in the samples containing 5 mg and 10 mg of glucose after 288 and 336 hours (Table 1B—Part I) as compared to the sample containing no glucose can only be interpreted on the basis that glucose stimulates the growth and multiplication of bacteria to a larger extent as a result of which the sugar containing culture solution comes to have a larger number of bacteria as compared to the sample with no sugar. Hence, after the sugar has been consumed this solution, because of the larger number of bacteria present in it shows a more vigorous nitrite formation. Similar results have been obtained with other sugars also, as will be clear from the tables. The optimum quantity of each sugar to produce enhanced growth of the bacteria may however vary. The time required to produce higher concentration of nitrite as compared to the sample containing no sugar may also vary with different sugars. Such a difference is natural and may be ascribed to the difference in the nature and complexity of the sugars.

The results of our study with larger concentrations of glucose (from 100 mg to 1000 mg/82 ml) are shown in Table 11. These results indicate that as the concentration of the sugar is increased the nitrite formation is not only delayed as with smaller concentrations but is also retarded.

The amount of sugar which is consumed by the bacteria when a high concentration (*e.g.* 1000 mg/82 ml) of glucose has been taken initially is very small even in 840 hours. This consumption is insignificant for such a long period. From the above results the natural inference is that higher concentrations of sugars are harmful for the bacteria. They depress the growth. However, it has been found that the bacteria do not get destroyed by these higher concentrations of glucose even after 840 hours, for when from this solution an inoculation is made to fresh medium devoid of any organic substance the bacteria begin to grow again. This result is in agreement with the observation of Ruban[8] who found that large concentrations of organic compounds halt the nitrification but the cells remain viable and resume active nitrification when transferred to inorganic nutrients. It is thus clear that in the presence of larger amounts of sugars and other organic compounds, it is only the activity of the bacteria that is affected and they do not get killed ; they remain in the dormant state and when favourable condition is restored they begin to grow and perform their activity again.

The results contained in Table 9B indicate that the effect of mannitol on nitrite formation is similar to that of sugars. Tartaric acid however has been found to be better as compared to these sugars in certain respects (Table 10). Unlike sugars, in the presence of tartaric acid nitrite formation takes place from the very beginning, both when the acid is present in smaller or in larger concentrations, the concentration employed being 5 mg to 100 mg/82 ml. Warington had also made the same observation. He had found that the nitrification of ammonium sulphate solution inoculated with soil took place in the presence of sodium potassium tartarate also, though the extent of nitrification was small. He further observed that when this tartarate was replaced by sugars the amount of nitrification was reduced thereby showing that the tartarate was better than sugars.

Thus, on the basis of our experimental results it can be concluded that sugars and other food material are not toxic or harmful to *Nitrosomonas* when supplied in proper amounts. Instead they serve as food material for the organism in much the same way as they serve for other organisms. It is only a question of the proper amount. When large amounts of these substances are supplied they do produce harmful effect. With other living organisms including higher ones also we find more or less the same thing. A food material taken in quantities much in excess of the requirements very often produces ailments in higher organisms and some times even this may cause death. Similarly, with *Nitrosomonas* sugars and other organic food material no doubt serve as food but if they are taken in excessively large amounts they produce harmful effects and thereby the growth and activity of the organism are depressed.

Our view about the non-toxicity of organic food material to *Nitrosrmonas* receives support from the observations of certain other workers also. In 1914 Kossowiez[9] had observed that carbohydrates were useful and stimulated the process of nitrification in soil. Kingma Boltjes[10] found that Narshtoff Hyden (a peptone product) stimulated the growth of the colonies of the nitrite formers. Similar results had been obtained by Fred and Davenport[11] and Hes[12]. Kingma Boltjes had also performed experiments to confirm if glucose was toxic to *Nitrosomonas* as had been reported by Winogradsky and subsequently supported by others but he failed to get any such confirmation. Bomecke[13] reported that the presence of 0·1% nutrient broth, 0·66% yeast extract or 0·05% peptone did not produce any adverse effect upon the process of nitrification. Quastel[14] observed that M/200 glucose only delayed the process of nitrification but did not affect

the growth. Jensen[15] has recently shown that the oxidation of ammonium sulphate to nitrite could go on even in the presence of high concentration of organic matter like 0·37M sodium formate, 0·44M sodium acetate, 0·30M succinate, 0·56M glycerol, 0·56M glucose and 0·58M urea. Buswell[16] has produced evidences that glucose in concentration up to 200 p.p.m. does not inhibit the growth of nitrifiers. The work of Kalinenko[17] showing that the presence of organic food material in the culture medium of *Nitrosomonas* increases the bacterial proteinous mass also lends support to our view that the organic food materials instead of causing any toxicity produce beneficial effect when they are snpplied in proper amounts.

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RATIONALS VERSUS IRRATIONALS

By

D. RAMESWAR RAO

Department of Mathematics, Osmania University, Hyderabad

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This paper gives a method of provin? that for a type of expressions in a given triad of rational integers of same parity in arithmetic progression, there exist a same type of expressions of equal valu‹s in a unique triad of irrational integers in arithmetic progression.

Theorem.—For any triad of rational integers ξ, η, ζ of same parity, in arithmetic progression; there exists uniquely a corresponding triad of irrational integers λ, μ, ν, in arithmetic progression such that

$$\left.\begin{aligned} \xi(\xi+\eta) &= \lambda(\lambda+\mu) \\ \eta(\xi-\eta) &= \mu(\lambda-\mu) \\ \xi^2+\eta^2 &= \lambda^2+\mu^2 \\ 2\xi\eta &= \lambda^2-\mu^2 \\ \xi^2-\eta^2 &= 2\lambda\mu \\ \zeta(\zeta+\eta) &= \nu(\nu+\mu) \\ \eta(\zeta-\eta) &= \mu(\nu-\mu) \\ \zeta^2+\eta^2 &= 2\nu\mu \\ 2\zeta\eta &= \zeta^2-\mu^2 \\ \zeta^2-\eta^2 &= 2\nu\mu \end{aligned}\right\} \tag{A}$$

Proof.—Fi1st consider

$$\left.\begin{aligned} x(x+y) &= X(X+Y) \\ y(x-y) &= Y(X-Y) \end{aligned}\right\} \tag{I}$$

Let $X = ux$, $Y = vy$

Then from I, we have

$$\left.\frac{x}{y} = \frac{uv-1}{1-u^2} = \frac{1-v^2}{1-uv}\right\} \tag{II}$$

so that

$$\left. u = \frac{-v \pm \sqrt{2}(v^2-1)}{1-2v^2}\right\} \tag{III}$$

From III and II it is obvious that x, y become rational and X, Y become irrational if and only if v is of the form $\sqrt{2}k$, for k rational ($= \frac{l}{m}$,) say where l and m are any integers)

Then
$$v = \sqrt{2}\frac{l}{m}$$

$$\therefore u = \sqrt{2}\frac{P_i}{Q_i} \; (i = 1, 2)$$

where

$$\left.\begin{array}{l} P_1 = l + m \\ P_2 = l - m \\ Q_1 = m + 2l \\ Q_2 = m - 2l \end{array}\right\} \quad \text{(IV)}$$

Therefore the only solutions of I such that x, y are rational integers and X, Y are irrational integers are

$$x = Q_i, y = m, X = \sqrt{2}\, P_i, Y = \sqrt{2}\, l \; ; i = 1, 2 \quad \text{(V)}$$

It can be very easily verified that the equations

$$x^2 + y^2 = X^2 + Y^2$$
$$2xy = X^2 - Y^2$$
$$x^2 - y^2 = 2XY$$

are simultaneously satisfied for the same value of x, y and X, y given by V and IV

$$\left.\begin{array}{l} \text{Hence the complete solution of equations (A) such that } \xi, \eta, \zeta \text{ are} \\ \text{rational and } \lambda, \mu, \nu \text{ are irrational is} \\ \xi = m - 2l, \quad \eta = m, \quad \zeta = m + 2l \\ \lambda = \sqrt{2}(l - m), \; \mu = \sqrt{2}\, l, \; \nu = \sqrt{2}(l + m) \end{array}\right\} \quad \text{(VI)}$$

where l and m are any integers

Now ξ, η, ζ are of same parity and η is the arithmetic mean of ξ and ζ

$$\left.\begin{array}{l} \text{Therefore if } \eta = m, \text{ we can find a rational integer } l \text{ such that} \\ \xi = m - 2l \text{ and } \zeta = m + 2l \end{array}\right\} \quad \text{(VII)}$$

Hence from VI and VII the theorem is proved

A NUMERICAL EXAMPLE

Putting $l = 2, m = 1$, we get

$$\xi = -3, \quad \eta = 1, \quad \zeta = 5,$$
$$\lambda = \sqrt{2}, \quad \mu = 2\sqrt{2}, \quad \nu = 3\sqrt{2}$$

Hence we have the following equations :

$$5(5 + 1) = 3\sqrt{2}(3\sqrt{2} + 2\sqrt{2})$$
$$1(5 - 1) = 2\sqrt{2}(3\sqrt{2} - 2\sqrt{2})$$
$$5^2 + 1^2 = (3\sqrt{2})^2 + (2\sqrt{2})^2$$
$$2 \cdot 5 \cdot 1 = (3\sqrt{2})^2 - (2\sqrt{2})^2$$
$$5^2 - 1^2 = 2(3\sqrt{2})(2\sqrt{2})$$
$$(-3)\{(-3) + (1)\} = (\sqrt{2})\{(\sqrt{2}) + (2\sqrt{2})\}$$
$$(-1)\{(-3) - (1)\} = (2\sqrt{2})\{(\sqrt{2}) - (2\sqrt{2})\}$$
$$(-3)^2 + (1)^2 = (\sqrt{2})^2 + (2\sqrt{2})^2$$
$$2(-3)(1) = (\sqrt{2})^2 - (2\sqrt{2})^2$$
$$(-3)^2 - (1)^2 = 2(\sqrt{2})(2\sqrt{2})$$

WAVES IN A STRATIFIED, COMPRESSIBLE, INVISCID FLUID CONFINED BETWEEN TWO RIGID BOUNDARIES

By

M. M. GUPTA *and* J. M. GANDHI

Department of Physics, University of Rajasthan, Jaipur

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ABSTRACT

The stability of a compressible, nonviscous fluid of density ρ varying in the direction of gravity, confined between two rigid boundaries and subjected to an initial infinitesimal perturbation is discussed. The solution is obtained in the form of integrals. The fluid is assumed to be stratified according to the law

$$\rho = \rho_0 \exp(\beta z)$$

where ρ_0 and β are constants. The stratification is assumed to be too small to consider any variation in the velocity of sound in the fluid. For all values of the wave lengths of the disturbance, hydrodynamic waves are produced and for stable stratification one gets undamped waves. The results are graphically illustrated.

1. INTRODUCTION

Vandervoort (1961) showed that a Variational principle characterising the equilibrium of a heavy, nonviscous and compressible fluid exists and indicated that it might have little importance because of the complicated dependence of growth rate n on the wave number of the disturbance k Moreover, he had discussed only the case of two superposed fluids of constant densities and having constant value of the velocity of sound. Mitchner and Landshoff (1964) stated that Vandervoort's assumption of constant density and constant velocity of sound in the fluid appears to be unjustified. However, in the present paper, for the tractibility of the problem, the velocity of sound has been assumed to be constant though the density variation has been taken into account. This is justified on the basis that the stratification of density is assumed to be very small, *i.e.*

$$\beta d << 1 \tag{1}$$

Besides, since the velocity of sound varies inversely as the underroot of density for a small change in the density of the fluid, the corresponding change in the velocity of sound is still smaller.

2. THE BASIC EQUATIONS AND THE BOUNDARY CONDITIONS

The characteristic equation for the present problem obtained by Vandervoort (1961) is

$$n^2 D \left[\rho \left(k^2 + \frac{n^2}{V^2}\right)^{-1} Dw \right] - n^2 \rho w + gk^2 \left[\left(D + \frac{g}{V^2}\right) \rho \left(k^2 + \frac{n^2}{V^2}\right)^{-1} \right] w = 0 \tag{2}$$

where $$n = \frac{\partial}{\partial t}, \quad D = \frac{d}{dz} \text{ and } k^2 = k_x^2 + k_y^2 \tag{3}$$

k being the total wave number of the disturbance in the horizontal plane, g is the acceleration due to gravity acting vertically along z- axis, w is the z component of the perturbed velocity, and V is the velocity of sound in the fluid.

The solution to equation (2) has to be sought subjected to some boundary conditions. As in the present case, since the fluid terminates at the rigid boundaries, velocity $\vec{u}\,(u, v, w)$ can have no normal component at the boundary. Therefore

$$w = 0 \text{ at } z = 0 \text{ and } z = d \tag{4}$$

Although boundary condition (4) is the only condition that is to be satisfied, since the fluid is nonviscous, yet, to compare the results for the corresponding viscous case for two rigid boundaries, it will be adviseable to assume that

$$Dw = 0 \text{ at } z = 0 \text{ and } z = d \tag{5}$$

for two rigid surfaces seperated by a distance d.

It has been shown by Gupta and Gandhi (1964) that a variational principle exists for the present problem and thus the equation (2) admits a solution obtainable by a first order iteration procedure, thus enabling the determination of n as a function of wave number k. The characteristic equation reduces to

$$\left(\frac{N^2}{V^2} + Nk^2\right) I_1 + N I_2 - gk^2 I_3 = 0 \tag{6}$$

where

$$I_1 = \int_0^d \rho\, w^2\, dz \tag{7}$$

$$I_2 = \int_0^d \rho\, D^2 w\, dz \tag{8}$$

$$I_3 = \int_0^d \left(D + \frac{g}{V^2}\right) \rho\, w^2\, dz \tag{9}$$

and

$$N \equiv n^2 \tag{10}$$

3. EQUILIBRIUM OF THE FLUID

As the fluid is bounded by two rigid boundaries, we choose w as

$$w = A\left(1 - \cos \frac{\pi s}{d} \cdot z\right) \tag{11}$$

so that the boundary conditions given by equations (4) and (5) are satisfied, s being a positive integer. The stratification is assumed to obey the law

$$\rho = \rho_0 \exp\,(\beta z). \tag{12}$$

Measuring n and k in the dimensionless form

$$y = nd/V\sqrt{R} \tag{13}$$

$$x = kd/R \tag{14}$$

and introducing the dimensionless parameters.

$$R = \pi s \tag{15}$$

and

$$T = \frac{d^2}{V^2}\,(g\beta + g^2/V^2) \tag{16}$$

equation (6) reduces to

$$3y^4 + y^2 (3x^2 + 1) R - 3x^2 T = 0 \tag{17}$$

giving

$$y = \frac{1}{\sqrt{6}} \left[\pm \sqrt{(3x^2 + 1)^2 R^2 + 36x^2 T} - (3x^2 + 1) R \right]^{\frac{1}{2}} \tag{18}$$

Case I (T = 0).

Taking negative sign before the radical in equation (18), y becomes imaginary and the fluid undergoes undamped oscillations with angular frequency $I(y)$, wave velocity Vw and group velocity Vg respectively given by :

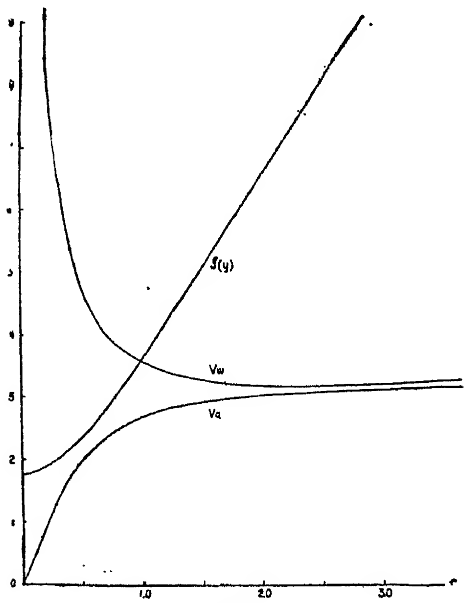
$$I(y) = \left[R(3x^2 + 1) \right]^{\frac{1}{2}} \tag{19}$$

$$Vw = \frac{I(y)}{x} = \frac{1}{x} \left[R(3x^2 + 1) \right]^{\frac{1}{2}} \tag{20}$$

and

$$Vg = \frac{d}{dx} I(y) = 3R/Vw \tag{21}$$

Figure 1 illustrates the variations in $I(y)$, Vw and Vg with x for $R = \pi$ ($s = 1$) and $T = 0$.



x ⟶

Fig. 1

The group velocity Vg, wave velocity Vw and angular frequency $I(y)$ as a function of the wave number x for $T = 0$ and $R = \pi$ ($s = 1$) according to equations (19) to (21).

Case II ($T > 0$)

(a) Taking positive sign before the radical y becomes real and the fluid is aperiodically damped with the damping coefficient

$$y = \frac{1}{\sqrt{6}} \left[\sqrt{R^2(3x^2+1)^2 + 36x^2 T} - R(3x^2+1) \right]^{\frac{1}{2}} \tag{22}$$

(b) If we take negative sign before the radical in equation (18), y becomes imaginary leading to the propagation of hydrodynamic waves with angular frequency wave and group velocities $I(y)$, Vw and Vg respectively given by

$$I(y) = \left[\tfrac{1}{6} \left\{ R(3x^2+1) + \sqrt{R^2(3x^2+1)^2 + 36x^2T} \right\} \right]^{\frac{1}{2}} \tag{23}$$

$$Vw = \frac{1}{x} \left[\tfrac{1}{6} \left\{ R(3x^2+1) + \sqrt{R^2(3x^2+1)^2 + 36x^2T} \right\} \right]^{\frac{1}{2}} \tag{24}$$

and

$$Vg = \frac{1}{2Vw} \left[R + \frac{R^2(3x^2+1) + 6T}{\sqrt{R^2(3x^2+1)^2 + 36x^2T}} \right] \tag{25}$$

Figure 2 depicts the variations of $I(y)$, Vw and Vg as a function of x for $R = \pi$ ($s = 1$) and $T = 1$.

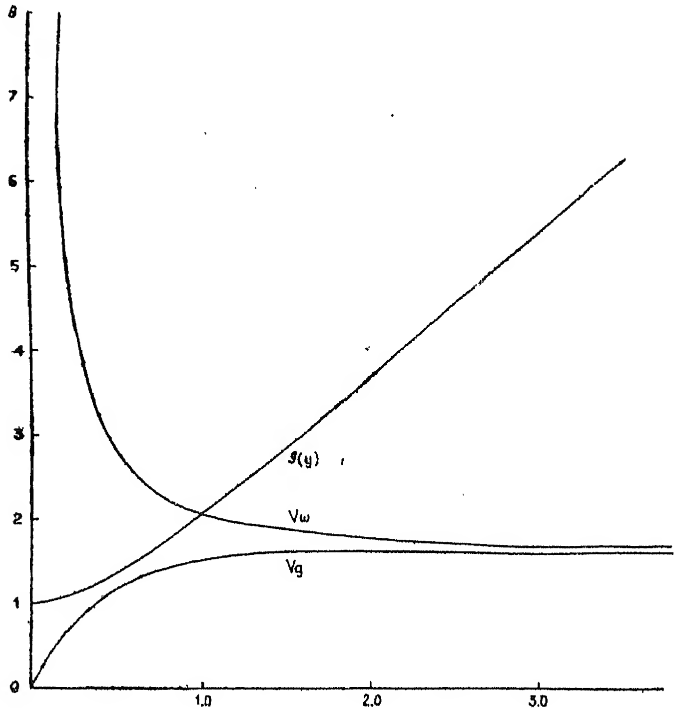


FIG. 2

The group velocity Vg, wave velocity Vw and angular frequency $I(y)$ for $T = 1$ and $R = \pi$ ($s = 1$) as a function of the wave number x according to equait.ons (23) to (25).

From case II (*b*) we conclude that we have undamped waves while from case II (*a*) we have aperiodic damping for the same wave numbers of the disturbance, which is physically unjustified. Since the fluid is inviscid, there can be no damping and as such case II (*a*) has to be rejected on physical grounds.

Case III ($T < 0$, $T = -T_1$; T_1 being positive).

This is the case of stable stratification.

(*a*) Taking positive sign before the radical in equation (18) y becomes imaginary, we again have waves with angular frequency $I(y)$ wave and group velocities Vw and Vg respectively given by

$$I(y) = \left[\tfrac{1}{6} \left\{ R(3x^2 + 1) - \sqrt{R^2(3x^2 + 1)^2 - 3\,6x^2\,T_1} \right\} \right]^{\frac{1}{2}} \tag{26}$$

$$Vw = \frac{1}{x} \left[\tfrac{1}{6} \left\{ R(3x^2 + 1) - \sqrt{R^2(3x^2 + 1)^2 - 3\,6x^2\,T_1} \right\} \right]^{\frac{1}{2}} \tag{27}$$

and

$$Vg = \frac{1}{2Vw} \left[R - \frac{R^2(3x^2 + 1) - 6T_1}{\sqrt{R^2(3x^2 + 1)^2 - 3\,6x^2\,T_1}} \right] \tag{28}$$

there being no damping of the disturbance. Figure 3 gives the variations in $I(y)$, Vw and Vg as a function of x for $T_1 = 1$ and $R = \pi$ ($s = 1$).

(*b*) Taking negative sign before the radical in equation (18) again there is no damping of the fluid but the waves are generated. The results are shown in

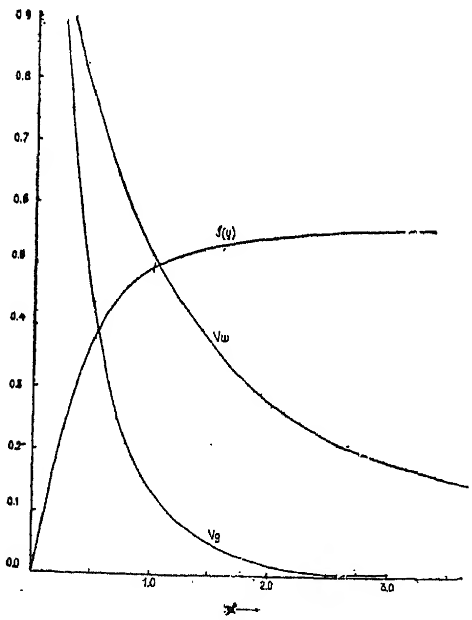


FIG. 3

The group velocity Vg, wave velocity Vw and angular frequency $I(y)$ for $T_1 = 1$ and $R = \pi$ ($s = 1$) as a function of the wave number x according to (26) to (28).

Figure 4 for $T_1 = 1$ and $R = \pi$ $(s = 1)$. The waves have angular frequency, wave and group velocities $I(y)$, Vw and Vg respectively as

$$I(y) = \left[\tfrac{1}{6} \left\{ R(3x^2 + 1) + \sqrt{R^2(3x^2 + 1)^2 - 3\,6x^2 T_1} \right\} \right]^{\frac{1}{2}} \tag{29}$$

$$Vw = \frac{1}{x} \left[\tfrac{1}{6} \left\{ R(3x^2 + 1) + \sqrt{R^2(3x^2 + 1)^2 - 3\,6x^2 T_1} \right\} \right]^{\frac{1}{2}} \tag{30}$$

and

$$Vg = \frac{1}{2Vw} \left[R + \frac{R^2(3x^2 + 1) - 6T_1}{\sqrt{R^2(3x^2 + 1)^2 - 3\,6x^2 T_1}} \right] \tag{31}$$

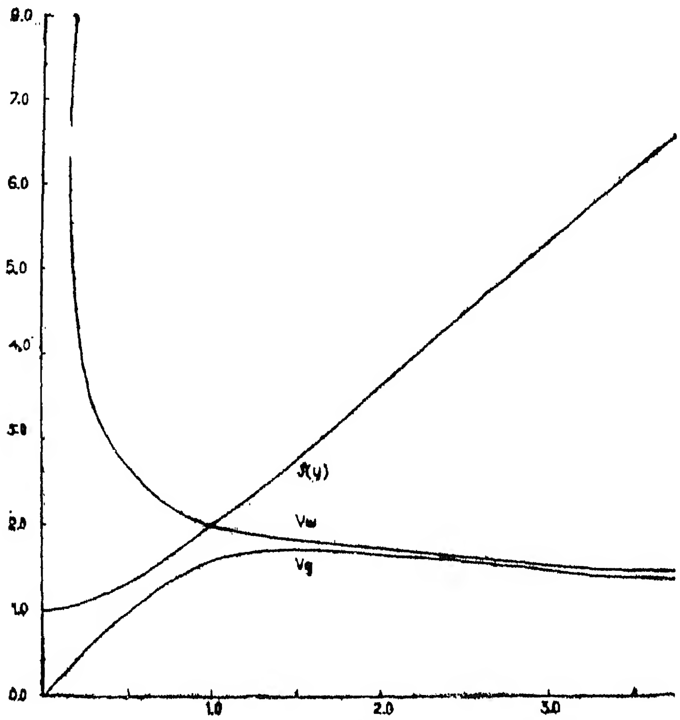


FIG. 4

The group velocity Vg, wave velocity Vw and angular frequency $I(y)$ as a function of the wave number x for $T_1 = 1$ and $R = \pi$ $(s = 1)$ according to equations (29) to (31).

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STUDIES ON THE FORMATION OF SOME TUNGSTO-CITRATES

By

SARJU PRASAD *and* LALLAN PRASAD PANDEY

Chemical Laboratories, Banaras Hindu University, Varanasi

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The formation of a number of tungsto citrates has been ascertained by conductance measurements. The compounds have been isolated in a few cases and probable structures have been assigned to them.

J. Lefort[1], for the first time obtained tungsto-citrates, and H. Grossmann and H. Krämerr[2] studied the conductivity of the compound $Na_2[WO_3(C_6H_6O_7)]$ at 25°C, at different dilutions. G. G. Henderson[3] *et al.* prepared $WO_2\ C_6H_6O_7.Na)_2.C_6H_7O_7Na\frac{1}{2}H_2O$; $WO_2(C_6H_6O_7K)_2.\ C_6H_7O_7K.3\frac{1}{2}H_2O$; $WO_2(C_6H_6O_7NH_4)_2.C_6H_7O_7.NH_4.H_22O$ by dissolving WO_3 in aqueous solutions of primary alkali citrates and crystallising the resultant solutions, and $2WO_2\ (C_6H_6O_7)_2Ba.(\ _6H_7O_7)_2Ba.10H_2O$ by the action of $BaCl_2$ on K-salt. Bobtelsky and Simchen[4] carried out conductometric titrations of sodium tungstate against tri sodium citrate in the presence and absence of H_2O_2. The reactions of sodium tungstate with some organic acids in aqueous solutions were studied spectrometrically by G. S. Savehenko[5]. G. S. Rao[6] studied the effect of mineral and some organic acids and certain salts on Na_2MoO_4 and Na_2WO_4 and observed that they form definite complex with WO_4^{--},MoO_4^{--} ions. E. Richardson[7] studied the complex formation between α-hydroxy acids and molybdic and tungstic acids by ion exchange method. Complex formation between sodium tungstate and malic acid has been studied in this laboratory[8] and the work is now extended to study the complex formation between sodium tungstate and citric acid.

EXPERIMENTAL

All the chemicals used wese of B. D.H. or Merck's "extra pure" quality. Conductivity water was used through out the titration procedure.

0·04032M sodium tungstate and 0·04998M citric acid solutions were prepared. The strength of sodium tungstate solution was determined by estimating tungsten as tungstic oxide in a known volume of the solution and that of citric acid solution by titrating it against standard solution of sodium hydroxide.

5 ml of sodium tungstate solution (0·004032M) diluted to 65 ml was placed on a magnetic stirrer and the dip cell properly dipped in solution. The titration cell was immersed in a thermostat to control the temperature within $\pm$ 0·5°C. The conductance was measured by Mullard conductivity bridge type E 7566 with direct reading visual balance detector. 0·1 ml citric acid solution was run down every time from a micro-burette. The solution was stirred well and the conductance recorded.

In the reverse titration 5 ml citric acid (0·004998M) was taken in the cell and titrated against sodium tungstate (0·04032M). Conductance was plotted against volm. of citric acid (*cf.* Fig. I, Curve I) and sodium tungstate (*cf.* Fig. I, Curve II). The equivalence points were taken as the points of intersection of the two portions of the curves. Direct as well as reverse titrations indicated the formation of two types of complexes, one in molar ratio 1 : 1 and other in 1 : 2 of sodium tungstate and citric acid.

S.No.	Compound	Colour	%W Found	%W Calc.	%Other Metal Found	%Other Metal Calc.	%Carbon Found	%Carbon Calc	%Hydrogen Found	%Hydrogen Calc.
1.	$Na_2[WO_2(C_6H_6O_7)_2]2H_2O$	White	26·91	27·14	6·73	6·79	21·40	21·24	2·48	2·36
2.	$Pb[WO_3.C_6H_6O_7]4H_2O$	,,	26·02	26·24	28·49	29·55	10·03	10·27	2·10	2·00
3.	$Ag_2[WO_3.C_6H_6O_7]3H_2O$	,,	26·15	26·59	30·98	31·22	10·24	10·40	1·81	1·73
4.	$Hg[WO_3.C_6H_6O_7]3H_2O$	,,	26·68	27·19	30·18	29·65	10·56	10·64	1·82	1·77
5.	$Pb[WO_3.C_6H_6O_7]H_2O$	,,	29·03	28·43	31·16	32·02	10·97	11·13	1·33	1·24
6.	$Ag_2[WO_3.C_6H_6O_7]H_2O$	,,	27·32	28·05	32·71	32·94	10·78	10·97	1·28	1.22
7.	$Hg[WO_3.C_6H_6O_7]H_2O$	,,	29·77	28·72	32·16	31·31	11·09	11·24	1·30	1·25
8.	$Cu[WO_3.C_6H_6O_7]nH_2O$	–	–	–	–	–	–	–	–	–
9.	$Mg[WO_3.C_6H_6O_7]nH_2O$	–	–	–	–	–	–	–	–	–
10.	$Zn[WO_3.C_6H_6O_7]nH_2O$	–	–	–	–	–	–	–	–	–
11.	$Cd[WO_3.C_6H_6O_7]nH_2O$	–	–	–	–	–	–	–	–	–
12.	$UO_2[WO_3.C_6H_6O_7]nH_2O$	–	–	–	–	–	–	–	–	–

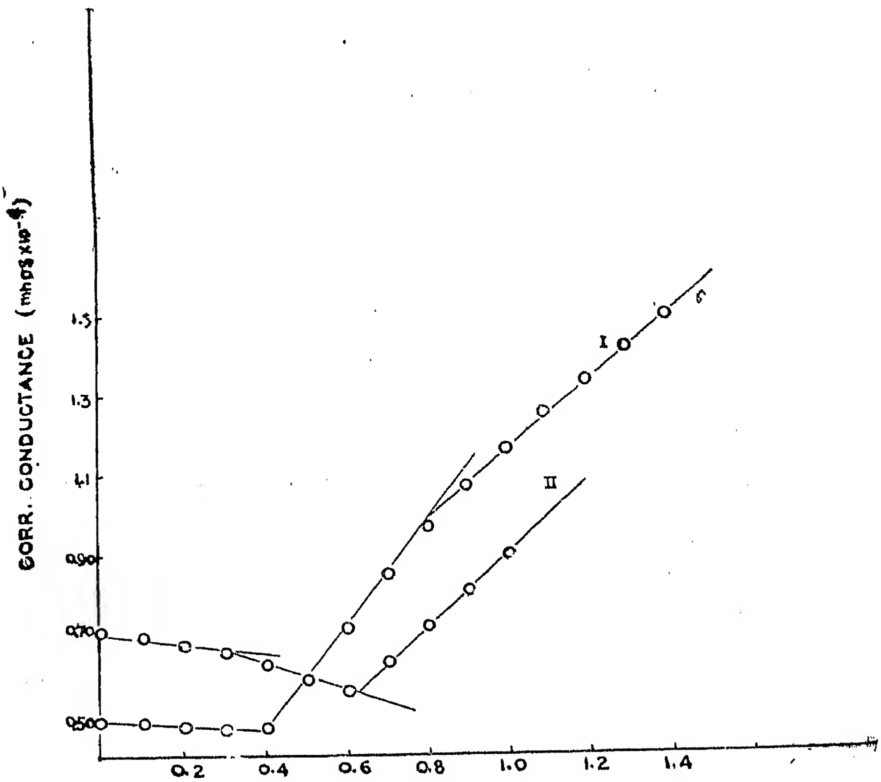


FIG. 1

Isolation of Compounds :

Solutions of sodium tungstate and citric acid were mixed in molar ratio 1 : 1. The mixture on evaporation gave a pasty mass which could not be crystallised. Aqueous solution of the pasty mass, on treatment separately with $Pb(NO_3)_2$, $AgNO_3$ and $Hg(NO_3)_2$ gave lead, silver and mercury tungsto citrates, respectively, in which the ratio of tungsten to citrate ion was found to be 1 : 1.

When a mixture of solutions of sodium tungstate and citric acid in molar ratio 1 : 2 was evaporated and treated with alcohol, a crystalline product was obtained. It was dissolved in water and reprecipitated by alcohol, filter pressed and finally dried over $CaCl_2$. On analysis it was found as $Na_2[WO_2(C_6H_6O_7)_2].2H_2O$. Lead, silver and mercury tungsto citrates were precipitated from the solution of $Na_2[WO_2(C_6H_6O_7)_2].2H_2O$ by adding $Pb(NO_3)_2$ $AgNO_3$ and $Hg(NO_3)_2$ respectively. It is interesting to note that in all the cases compounds were formed, in which tungsten and citrate ion are in molar ratio of 1 : 1 as in compounds obtained previously. However, these compounds differ from the previous ones in having less water of crystallisation.

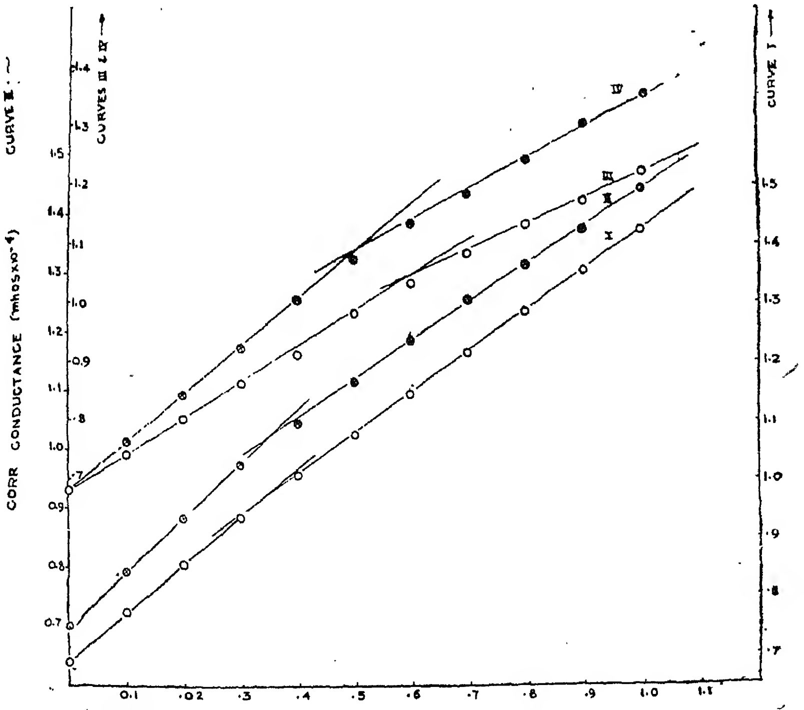


FIG. 2

Attempts were made to prepare tungsto-citrates of Cu, Zn, Cd, Mg and UO_2 but these could not be isolated, probably due to their solubility in water. Their formation was studied by conductometric method.

5 ml solution of sodium tungsto-citrate (0·002708M) diluted to 65 ml, was each time titrated separately against copper sulphate (0·02818M) Fig. 3, Curve I silver nitrate (0·04701M) Fig. 2 Curve III, magnesium sulphate (0·03702M) Fig. 2 Curve I, zinc sulphate (0·03930M) Fig. 2 Curve II, cadmium sulphate (0·03651M)

Fig. 3 Curve III, mercuric nitrate (0·03581M) Fig. 3 Curve II, lead nitrate (0·02612M) Fig. 2 Curve IV, and uranyl nitrate (0·04066M) Fig. 3 Curve IV.

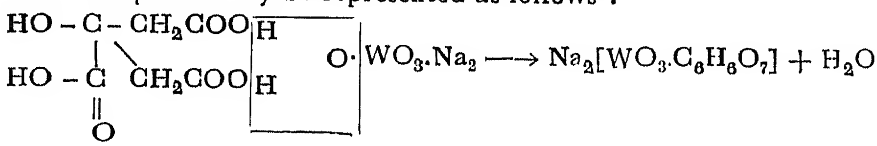
An examination of the curves shows the formation of corresponding salts in which the molar ratio of sodium tungsto-citrate to the metallic salt is 1 : 1 except in case of silver where the ratio is 1 : 2.

Analysis.—Tungsten was estimated as tungstic oxide after precipitating it with dil. HNO_3 and subsequent ignition, lead as sulphate, mercury as sulphide, silver as chloride. Carbon and hydrogen were estimated by micro-combustion method and water of crystallisation and organic matter were calculated from the data. Sodium was found by difference.

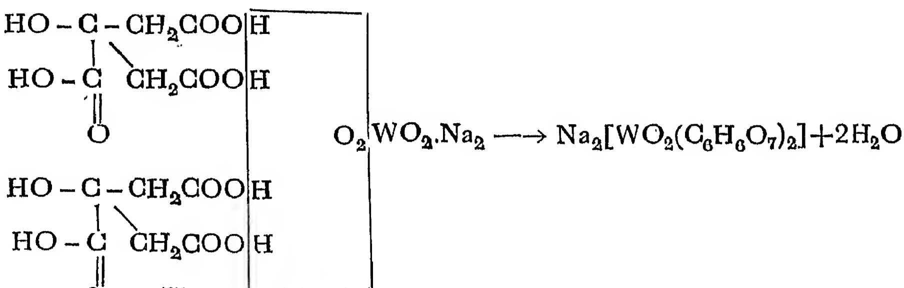
Properties.—The compounds are stable and white in colour. Sodium salts are highly soluble in water but insoluble in alcohol. Water solution is acidic.

DISCUSSION

It is evident from Fig. 1 (curves I - II) that sodium tungstate forms two types of complexes with citric acid in the molar ratios 1 : 1 and 1 : 2. The structure of the complexes may be represented as follows :

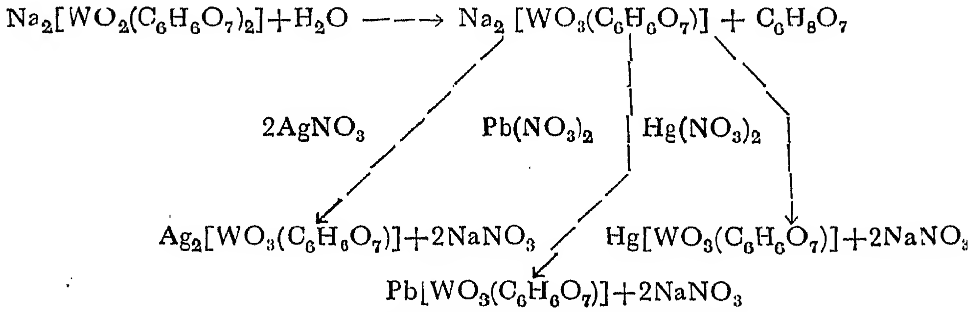


I



II

But the later compound when treated with silver lead and mercury nitrates gives compounds having one citrate ion in the molecule. This shows that when $Na_2 [WO_2(C_6H_6O_7]_2 \, 2H_2O$ dissolves in water, it gets dissociated and the following reactions take place



The above assumption is also supported by the fact that the solution of the sodium compound is strongly acidic.

Is is evident from conductometric titrations that $Na_2[WO_2(C_6H_6O_7)_2]$ forms compounds with Cu(ic), Zn, Cd, Mg and UO_2 in the ratio of 1 : 1. The compounds may be represented as $M[WO_3(C_6H_6O_7)]$ similer to compounds of Pb, Ag and Hg which have been isolated.

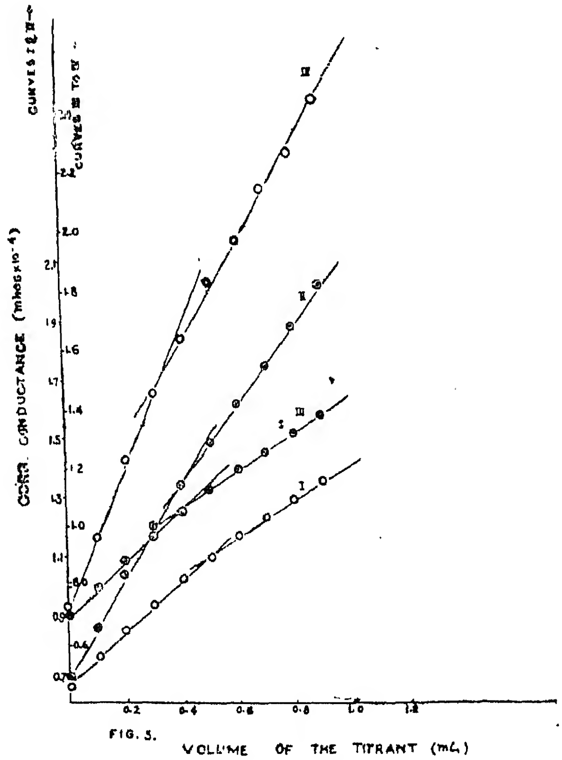


FIG. 5.

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SPECTROPHOTOMETRIC STUDIES ON THE TITANIUM (III) CHLORIDE-*P*-NITROBENZENE AZOCHROMOTROPIC ACID COMPLEX

By

M. MAHFOOZ KHAN *and* NASEER AHMAD

Department of Chemistry, Aligarh Muslim University, Aligarh

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ABSTRACT

Chromotrope 2B gives a pink coloured solution in water and forms a less pink complex with Titanium (III) chloride. The λ max for the ligand as well as the complex is 520 mμ. Beer and Lambert's Law is found to hold good at the concentration range of $0{\cdot}375\times10^{-3}$ — $0{\cdot}5\times10^{-4}$M. Job's method of continued variations in aqueous solution and in neutral pH buffer yields a molar ratio of 1 : 2 (Titanium (III) chloride : Chromotrope 2B). The same ratio emerges from molar-ratio method. The complex is stable between pH range of 2 to 9 while above this range precipitation occurs on keeping for sometime. A structure for the complex has been proposed.

Chromotropic acid is well known for its chelating properties and has been extensively studied. Comparatively very little work has been done on metal chelates of *p*-nitrobenzene azochromotropic acid (sodium salt) known as chromotrope 2B (abbreviated in this paper as CTB). Titanium (III) chloride forms a complex with chromotrope 2B, which is less pink than the chelating agent itself. The present communication deals with our spectrophotometric studies on this complex.

EXPERIMENTAL

Aqueous solution of titanium (III) chloride was prepared by dissolving crystals[1] of $TiCl_3.6H_2O$ in air free double distilled water and standardised.[2] Always fresh solutions were prepared before use and kept covered with a layer of kerosene oil throughout the investigations. B. D. H. reagent grade chromotrope 2B was used for preparing its solution.

For spectrophotometric measurements Bausch and Lomb spectronic -20 was used. Titanium (III) chloride and chromotrope 2B were mixed in the molar ratio of 1 : 1, 1 : 2 and 2 : 1 and solutions of different concentrations were prepared by diluting with air free double distilled water, to verify the applicability of Beer and Lambert's Law. It was found to hold good for the concentration range of $0{\cdot}375\times10^{-3}$ to $0{\cdot}5\times10^{-4}$M for 1 : 2 complex.

The method of Vosburgh and Cooper[3] was employed to ascertain the nature of complexes formed in aqueous solution. $0{\cdot}5\times10^{-4}$M solutions of titanium (III) chloride and chromotrope 2B were mixed in the ratio of 1 : 1, 1 : 2, 1 : 3, 2 : 1 and 0 : 1 and their absorbance was determined at different wave lengths in the visible range. The λ max. for the complex as well as chromotrope 2B is 520 mμ (Fig. 1).

$0{\cdot}8\times10^{-4}$M and $0{\cdot}6\times10^{-4}$M aqueous solutions of titanium (III) chloride and chromotrope 2B were mixed according to Job's method of continued variations[4] for two sets, their optical densities noted at 520 mμ and the 'Y' plotted against the composition of the mixture (Fig. 2).

FIG. NO 1

VOSBURGH AND COOPER METHOD

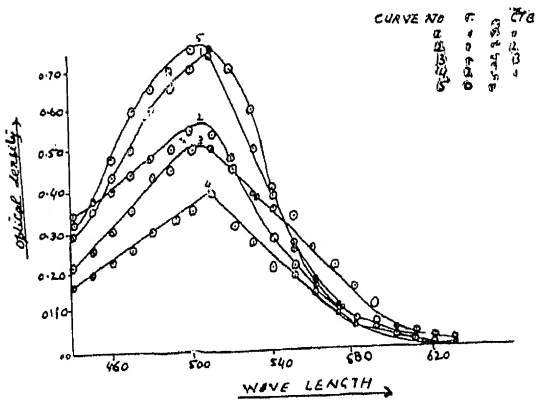


FIG. NO. 2

JOBS METHOD

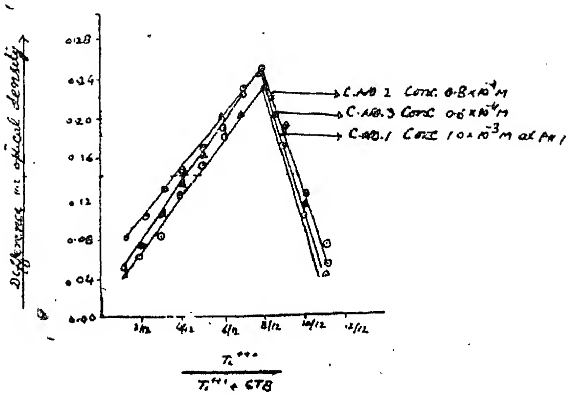


FIG. NO. 3.

MOLAR RATIO METHOD

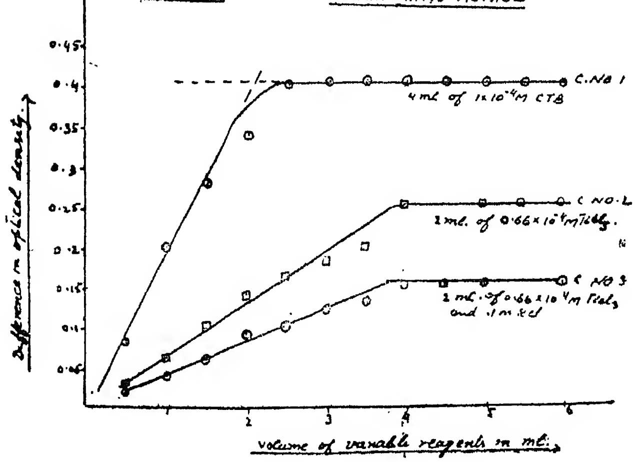
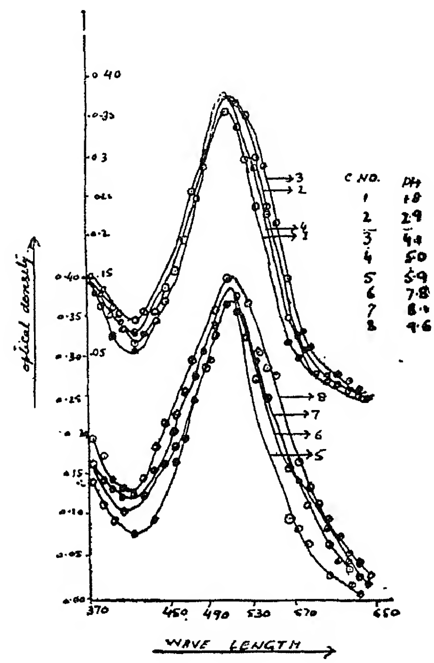


FIG. NO 4

EFFECT OF pH



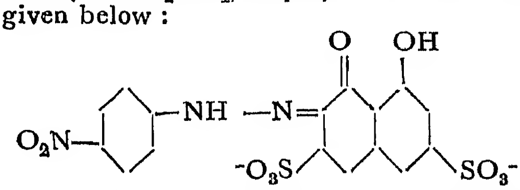
Molar ratio method[5] was also employed to ascertain the composition of the complex. In one set of experiments, the quantity of titanium (III) chloride was kept constant and that of chromotrope 2B was varied and in another set, the quantity of chromotrope 2B was constant and that of titanium (III) chloride was varied (Fig. 3).

To determine the effect of pH on the formation of the complex, equimolecular solutions (1×10^{-3}M) of the reactants were mixed in the ratio of 1 : 2 (metal : ligand) and then ten times diluted with buffer mixtures of pH values from 2 to 9. Above pH 9, precipitation occurs on keeping for sometime. The optical densities were determined at different wave lengths in the visible range. It was found that maximum absorption occurs at 520 mμ at all pH values. (Fig. 4).

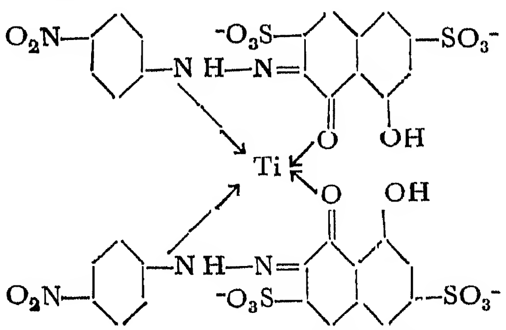
The complex was studied at neutral pH buffer by Job's method (Fig. 2) and at constant ionic strength of 0·1M KCl by molar ratio method (Fig. 3).

DISCUSSION

Job's method in aqueous solution and in neutral pH buffer gives the molar ratio of 1 : 2 for titanium (III) chloride and chromotrope 2B. This ratio is confirmed by molar ratio method performed in aqueous solution and at constant ionic strength. Sommer *et al*,[6] suggested that chromotrope 2B in aqueous solution and in acidic medium (0·1N H_2SO_4) displays maximum absorption at 517 mμ and exists in the form given below :



Since in our case the medium was neutral, the following structure may be suggested for the chelate.



ACKNOWLEDGEMENT

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STUDIES IN DRY MATTER AND CRUDE PROTEIN CONTENT OF OATS AT DIFFERENT STAGES OF GROWTH

By

HARI SHANKER *and* RAJ NARAIN PANDEY

Department of Agricultural Chemistry, Govt. Agricultural College, Kanpur, U. P.

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ABSTRACT

Field trials were made to study the effect of different levels of nitrogen and phosphorus on dry matter and crude protein contents of two varieties of oats at different stages of growth. It was observed that dry matter percentage increased and crude protein percentage decreased with advance in maturity. The application of phosphorus has slight effect in increasing the percentage of dry matter and crude protein, whereas the higher doses of nitrogen increased the crude protein percentage but decreased that of dry matter The percentage of dry matter and crude protein was higher in the variety 'Green mountain' than that in 'Oat-11'.

Recent reports of greater returns in feeding value when oats are harvested as forage compared to ripe grain and straw have stimulated interest in this crop for silage, green fodder, dry fodder, and for other forage uses.

The purpose of this study was to obtain information on dry matter and crude protein contents of two varieties of oats fertilized at different levels of nitrogen and phosphorus and harvested at various developmental stages from the view point of forage and grain production. Such knowledge could be of value in determining the maximum utilization of the crops.

REVIEW OF LITERATURE

Differing views are found among reports relative to the best dose of nitrogen and phosphorus and best stage of oat development for maximum yield of dry matter and crude protein. Berry (1920) reported that percentage of dry matter increased and crude protein percentage decreased with an advancement in maturity. Similar observations were made by Sotola (1937). Smith and Robb (1943) observed that dry matter of whole plant increased from 20·5–47·9% in case of control plot and 18·4–50·9% in the plot receiving extra nitrogen. He further reported that protein decreased steadily from 10·5–4·9% (control) and 13·3–6·0% (extra dressing), but protein content markedly increased at all the stages of growth by extra dressing. Nicholson (1957) reported that dry matter and crude protein contents were highest at the stage of ripening of the crop. Burton and Prime (1958) reported that application of nitrogen and phosphorus increased the protein content of oat crop. Sorensen (1959) stated that application of nitrogen and phosphorus in combination increased the dry matter, yield and total nitrogen both in straw and grain. Smith (1960) observed that dry matter percentage increased, while protein percentage declined with maturity. Stall-Cup (1960) found a negative correlation between dry matter and crude protein contents of oat, when nitrogen was applied at the rate of 20, 40 and 60 lbs. per acre. Similar results were also obtained by Wedin (1962), Maclaren and Garson (1958), and Meyer, Weir, Jones, and Hull (1957). Schmidt (1962) reported that highest yields of both nitrogen and dry matter were obtained when oats were harvested at maturity as grain and straw. He also found a marked

variation in nitrogen and dry matter contents due to different levels of nitrogen and also due to different varieties. Thatcher and Arny (1917), and Wiggans and Frey (1956) reported increased nitrogen content of grain from added nitrogen fertilizer.

METHOD AND MATERIAL

The experiment was conducted at the Student's Instructional Farm, Govt. Agricultural College, Kanpur, during Rabi season, 1964–65. Two varieties of oats, *i.e.* 'Oats–11' and 'Green–mountain' were sown at the rate of 30 kgms. per acre on 10th November, 1964. The design of the experiment was split-plot with 4 replications in which phosphorus was the main factor with two levels, *i.e.*, 0 and 30 lbs. per acre. It was applied at the time of sowing the crop. Nitrogen with four levels, *i.e.*, 0, 30, 60 and 90 lbs. per acre along with two varieties were the sub-factors. The combinations of nitrogen and varieties were applied in the sub-plots of the main plots with the help of random numbers with a view to know the best variety and suitable dose of nitrogen. Half of the amounts of nitrogen were applied as basal dressings, and rest were top-dressed with irrigation, 60 days after sowing.

The composite samples of four plots of similar treatment combinations from each block were taken at an interval of 21 days. The total number of samplings were 7 during whole period of crop growth. In the last sampling, grain and straw were studied separately.

Dry matter was determined by putting the harvested samples in an oven at 60°–70°C and dried to a constant weight. Nitrogen determinations were made by the Kjeldahl's method after samples were ground through a 20 mesh screen and dried at 70°C, as described by Piper (1950). Crude protein was obtained by multiplying the N–percentage of the factor 6·25.

RESULTS AND DISCUSSION

A description of the 7 stages of growth and the average percentage of the dry matter is given in table 1. Similarly the average crude protein percentage is given in table 2.

The dry matter percentage increased with advance in maturity of the crop in all the cases, *i.e.*, 13·1–88·14%. The straw contained higher percentage of dry matter than grain, *i.e.*, 89·22 and 84·63% respectively. The percent dry matter did not change greatly during the period of early growth until the heads were emerging from the boot (stage 4). Thereafter the percentage dry matter increased markedly until the maturity. Similar results were reported by Berry (1920), Sotola (1937), Nicholson (1957) and Smith (1960).

The dry matter percentage is markedly affected by the nitrogen and phosphorus fertilizaton. Application of phosphorus @ 30 lbs. per acre increased the dry matter percentage over control at all the stages of crop growth. Application of various levels of nitrogen, *i.e.*, 0, 30, 60 and 90 lbs. per acre are negatively correlated with the dry matter percentage. But there is a significant increase in the total yield of dry matter per acre upto the level of 60 lbs. per acre. Similar observations were made by Smith and Robb (1943), Sorensen (1959) and Schmidt (1962). The variety 'Green-mountain' had slightly higher percentage of dry matter than 'Oat–11' at all the stages of harvest.

TABLE 2

Average percentage of crude protein as affected by various levels of Phosphorus, Nitrogen and Variety at different stages of growth

Sl. No. of sampling		1	2	3	4	5	6	7		
Age of the crop in days		21	42	63	84	105	126	147		
Stage of growth		+ leaves stage	6–7 leaves stage	Boot stage	Heading stage	Early milk stage	Early dough stage	Harvesting stage		
Parts analysed		Whole plant	Whole plant	Whole plant	Whole plant	Whole plant	Whole plant	Grain	Straw	Whole plant
Phosphorus Levels	P_0	21·95	18·82	14·09	8·88	7·35	4·02	10·91	2·19	3·96
	P_{30}	22·14	19·66	14·33	9·16	7·57	4·34	11·02	2·42	4·04
Nitrogen Levels	N_0	21·47	17·70	12·61	7·64	5·92	3·44	10·04	1·45	3·16
	N_{30}	22·09	18·48	13·91	8·89	7·18	4·14	10·98	2·26	3·86
	N_{60}	22·26	19·59	14·62	9·53	7·99	4·54	11·37	2·75	4·40
	N_{90}	22·37	21·42	15·29	9·61	8·72	4·61	11·47	2·78	4·53
Varieties	Oat–11	22·02	19·14	14·12	8·95	7·41	4·12	10·93	2·26	4·06
	Green-mountain	22·07	19·36	14·29	9·03	7·51	4·24	11·00	2·35	3·97
	Average	22·05	19·27	14·16	8·96	7·46	4·18	11·96	2·37	4·00

TABLE 1

Average percentage of Dry matter as affected by various levels of Phosphorus, Nitrogen and Variety at different stages of growth

Sl. No. of sampling		1	2	3	4	5	6	7		
Age of the crop in days		21	42	63	84	105	126	147		
Stage of growth		4 leaves stage	6–7 leaves stage	Boot stage	Heading stage	Early milk stage	Early dough stage	Harvesting stage		
Parts analysed		Whole plant	Whole plant	Whole plant	Whole plant	Whole plant	Whole plant	Grain	Straw	Whole plant
Phosphorus	P_0	12·75	17·17	20·39	22·42	28·28	42·78	85·49	89·12	87·83
Levels	P_{30}	13·47	19·51	21·69	25·58	29·86	45·85	84·40	89·37	88·39
Nitrogen	N_0	13·83	20·52	22·25	26·07	30·35	46·53	85·16	90 10	88·72
Levels	N_{30}	13·32	17·93	21·40	25·17	29·85	43·35	84·68	89·59	89·40
	N_{60}	13·30	17·36	20·60	23·40	28·56	43·03	83·95	89·03	88·02
	N_{90}	12·36	15·87	18·90	23·42	26·58	40·62	83·50	88·28	87·48
Varieties	Oat–11	12·93	17·43	19·69	24·30	26·88	43·06	84·27	89·21	86·96
	Green-mountain	13·29	18·92	21·39	25·14	30·42	43·53	85·62	89·29	88·32
	Average	13·12	18·09	20·79	24·81	28·85	43·59	84·63	89·22	88·14

In all the treatments and varieties the crude protein percentage declined with the maturity of crop *i.e.* from 22·05–4·00%. The percentage of crude protein was higher in grain than straw, *i.e.*, 11·96 and 2·37% respectively. The percent crude protein decreased rapidly during the period of early growth until the kernels were in early milk (stage 5), after which a very little change was observed. Similar results were reported by Berry (1920), Sotola (1937) and Smith, (1960).

Applications of various levels of nitrogen and phosphorus in combination or nitrogen alone increased the crude protein percentage of the oat at all the stages of growth. Similar results were obtained by Smith and Robb (1943), Burton and Prine (1958), Sorensen (1959), Thatcher and Arny (1956), and Wiggans and Frey (1956). The crude protein percentage of 'Green-mountain' was higher than 'Oat-11'.

Differences in the percentage of dry matter and crude protein at all the stages of growth due to difference in fertility levels are not significant, whereas differences due to date of sampling are significant. There is negative correlation between the dry matter and crude protein percentage. The results were supported by Stall-Cup (1960), Wedin (1962), Maclaren and Garson (1958), Schmidt (1962), and Mayer *et al.* (1957).

SUMMARY

Analysis of oat crop at different stages of growth indicated that dry matter percentage increased and crude protein percentage decreased with advance in maturity. Application of phosphorus increased the dry matter and crude protein percentages to very little extent. Application of higher doses of nitrogen increased the crude protein percentage and decreased the dry matter percentage. The percentage of dry matter and crude protein was higher in 'Green-mountain' than that in 'Oat-11'.

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INVESTIGATIONS ON SOME AMINO COMPLEXES OF COPPER (II) OXALATE PREPARATION, PRELIMINARY INVESTIGATION AND ABSORPTION SPECTRA

By

GOPAL NARAIN

Department of Chemistry, University of Lucknow, Lucknow,

[Received on 24th March, 1966]

ABSTRACT

Several tetraco-ordinated complexes of copper (II) oxalate with uni and bidentate amines have been prepared. Molecular formulae on the basis of percentages of constituent elements are $Cu(C_2O_4)$ (amine)$_2$ and $Cu(C_2O_4)$ (amine). Conductivity measurements show the compounds to be non-electrolytes. Molecular weight measurements confirm this. Visible absorption measurements show a single band in the 600-900 mμ region of the spectrum.

INTRODUCTION

A survey of the literature shows that a large number of copper complexes with ammonia and various amines have been studied. However, only a few solid complexes have been prepared with aliphatic amines as ligands[1,2]. Although an extensive series of investigations with inorganic salts of copper are reported, comparatively very little work of this type has been done with the organic salts[3-6].

In the present paper the synthesis, some of the preliminary investigations and visible absorption measurements are reported.

EXPERIMENTAL

(A) Synthesis :

The following general method was used for the preparation of the complexes.

500 mgm of Copper (II) oxalate (AR/BDH) was suspended in 15 c.c. of acetone and was shaken with calculated quantity of amine for three hours. Resulting complex was filtered and dried over P_2O_5. The percentage composition of the individual complexes are given below.

(1) *Mono-oxalate, bis (ammine) Copper* (II)

Found Cu = 34·36%, N=14·98% ; $C_2H_6O_4N_2$ Cu requires Cu=34·24%, N=15·09%.

(2) *Mono-oxalato, bis (methylamine) Copper* (II)

Found Cu=29·43%, N=12·88% ; $C_4H_{10}O_4N_2$ Cu requires Cu = 29·74%, N= 13·11%.

(3) *Mono-oxalato, bis (ethylamine) Copper* (II)

Found Cu=26·49%, N=11·18% ; $C_6H_{14}O_4N_2$ Cu requires Cu=26·29%, N=11·59%.

(4) *Mono-oxalato, bis (n-propylamine) Copper* (II)

Found Cu=23·33%, N=10·06% ; $C_8H_{18}O_4N_2$ Cu requires Cu=23·56%, N=10·39%.

(5) *Mono oxalato, bis (iso-propylamine) Copper* (II)

Found Cu=23·40%, N=10·10% ; $C_8H_{18}O_4N_2$ Cu requires Cu=23·56%, N=10·39%.

(6) *Mono oxalato, bis (n-butylamine) Copper* (II)

Found Cu=21·03%, N=9·12%, $C_{10}H_{22}O_4N_2$ Cu requires Cu=21·34%, N=9·41%.

(7) *Mono-oxaloto, bis (iso-butylamine) Copper* (II)

Found Cu=21·04, N=9·20% ; $C_{10}H_{22}O_4N_2$. Cu requires Cu=21·34%, N=9·41%.

(8) *Mono oxalato, mono (ethylenediamine) Copper* (II)

Found Cu=29·98%, N=13·08% ; $C_4H_8O_4N_2$ Cu requires Cu=30·03%, N=13·24%.

(9) *Mono-oxalato, mono (propylenediamine) Copper* (II)

Found Cu=28·30%, N=12·21% ; $C_5H_{10}O_4N_2$ Cu requires Cu=28·16%, N=12·42%.

(B) Analysis and molecular weight measurements

Copper was estimated as copper salicylaldoxime complex and nitrogen by Kjeldahl method. Estimation of amine content and molecular weight measurement were done using the methods outlined by Padmaja Shukla and others.[7]

(C) Conductivity measurements.

Measurements were done in formamide and water at a concentration of 10^{-3}M. The cell used had a cell constant of 0·0245.

(D) Spectral measurements.

Spectra were recorded on a unicam S. P. 500 spectrophotometer using formamide as the solvent.

RESULTS

The percentages of constituents elements have been given along with the synthesis of complexes. The results of the determination of percentages of total bases, molecular weights, molar conductance, visible absorption spectra and colours are tabulated below.

No.	Formulae	Percentage of Base Calc.	Obs.	Molecular weight Calc.	Obs.	Molar conductance in mhos	Spectral Measurements Colour	λ max in mμ
1.	$[Cu(C_2O_4)(NH_3)_2]^\circ$	18·33	18·53	185·5	171	0·630	Blue	650
2.	$[Cu(C_2O_4)(CH_3NH_2)_2]^\circ$	29·04	29·34	213·5	202	0·341	,,	645
3.	$[Cu(C_2O_4)(C_2H_5NH_2)_2]^\circ$	37·27	37·60	241·5	230	0·912	,,	660
4.	$[Cu(C_2O_4)(n\text{-}pr)_2]^\circ$	43·78	43·52	269·5	254	0·901	,,	665
5.	$[Cu(C_2O_4)(n\text{-}bu)_2]^\circ$	49·08	49·28	297·5	308	0·725	,,	660
6.	$[Cu(C_2O_4)(eu)]^\circ$	28·37	28·60	211·5	199	0·812	Violet	660
7.	$[Cu(C_2O_4)(pn)]^\circ$	32·81	32·56	225·5	237	0·842	,,	648

In the above table n—pr represents n—propylamine
n—bu ,, n—butylamine
eu ,, ethylenediamine
pn ,, propylenediamine

DISCUSSION

On the basis of the percentages of metal, nitrogen and total bases, the molecular formulae turn out to be Cu(oxalate) $(amine)_2$ or Cu(oxalate) (amine) showing that two molecules of unidentate and one molecule of bidentate amine are added per molecule of Copper (II) oxalate. These complexes dissolve only in formamide and the solution has the typical ink blue colour characteristic of cupric amine complexes. The bidentate amine complexes are also soluble in water.

The complexes dissolve in formamide and water to give very dilute solution (Conc. 10^{-3}M) and the measurement of molar conductance of these solutions gives a value varying from 0·3 to 0·9 mhos. This value being less than one indicates, that these complexes behave as non-electrolytes[8]. When a voltage is applied to two platinum electrodes dipping in this formamide solution in a U tube, no migration of the blue colour towards the negative electrode was noticed, showing, thereby that the complexes do not ionise.

The freezing point determinations in formamide give the normal absolute value of molecular weight. These results suggest that oxalate ion is inside the co-ordination sphere, along with the two or one amine ligands. The formulae of the complexes must therefore be written as $[Cu(C_2O_4)\ (am)_2]^\circ$ and $[Cu(C_2O_4)\ (am)]^\circ$.

The arrangement of any four groups around any bivalent copper ion has usually been found to be planar, as tetrahedral disposition of the four valencies is extremely rare on the basis of crystal field theory[9]. The complexes exhibit a single absorption band in the range of 600–900 mμ. When a field is applied around any bivalent copper ion, the five 3d levels do not remain degenerate and split. The ground level in this case is associated with configuration $t_{2g}^6\ eg^3$ and excited level with $t_{2g}^5\ eg^4$. It can thus be concluded that on the exposure of light the positron which initially occupies a t_{2g} level comes to a eg level and thus a single band corresponding to this transition is to be expected, which is being observed in 600–900 mμ range of the spectrum.

The position of this band depends on the intensity of ligand field around metal. Thus replacement of water ligands in tetraquo copper (II) ion with amines changes the position of absorption band from 800 to 600 mμ for tetramino copper (II) ion. This bathochromine effect is due to the stronger ligand field environment produced around the metal.

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INFLUENCE OF LIGHT AND PHOSPHATE ON MINERALISATION OF NITROGEN AND FORMATION OF FREE AMINOACIDS DURING COMPOSTING OF WHEAT-STRAW

By

N. R. DHAR *and* S. P. JAISWAL

Sheila Dhar Institute of Soil Science, University of Allahabad

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ABSTRACT

Experimental results have been obtained showing the release of available nitrogen and formation of different aminoacids during the composting of wheat straw.
The presence of phosphates and light enhance these processes.

It is well known that the nitrogen of the organic matter in the protein type of compounds is released during the oxidation and decomposition of these carbonaceous compounds. Several intermediate nitrogenous compounds are produced which in course of time are oxidised to ammonia, nitrite and finally to nitrate. Dhar *et al*, (1930, 1933, 1934) have experimentally shown that ammonification and nitrification in soil at least in tropics, are predominantly photochemical processes. Ingham (1940) has remarked that although biological process generally considered to be the most important, non-biological nitrification brought about by sunlight is believed to be of special significance in tropics. Similarly, it has been reported that ammonia can be oxidized in soil to nitrate photochemically (Cultera *et al*, 1947) or physicochemically (Puri *et al*, 1946).

It is widely recognized that light plays an important role in the synthesis of a number of nitrogenous compounds some of which are of great biological interest. Baly *et al*, (1922, 1923) reported photosynthesis of aminoacids and nitrogen bases by the action of activated formaldehyde produced photochemically with nitrites. Dhar *et al*, (1935, 1955) obtained aminoacids by exposing to light solutions of nitrates with carbohydrates in contact with titania as photocatalyst.

In this paper we have investigated the photochemical action of light on the production of available nitrogen and formation of free aminoacids both in presence and absence of phosphates.

MATERIALS AND METHODS

100 gms. of well powdered wheat straw and 10 gms. of soil passed through 2mm. sieve were taken in clean enamelled dishes, 24 cms diameter. The phosphates were added at the rate of 0·5% P_2O_5. Two corresponding sets of experiments were started. One was exposed to the light of a 100 watt electric bulb with a reflector hung at a vertical distance of two feet above the table on which these dishes were placed. The other set of dishes containing in every detail the same materials were kept covered with a thick piece of black cloth. The moisture was maintained at 40% level and the contents were stirred on alternate days to facilitate the oxidation of organic substances.

After definite intervals of time representative samples were taken out, powdered, sieved and analysed on oven dry basis. Total nitrogen was determined by Kjeldahl salicylic acid reduction method, (John Brooks, 1936). Ammoniacal and nitrate nitrogen were determined by olsens method as modified by Richardson (Piper, 1942). Paper Partition Chromatographic techniques as developed by

Giri *et al*, (1952) and Consden *et al*, (1944) were employed for the identification of aminoacids. Quantitative estimations were carried out according to the colorimetric method of Harding and Mclean (1916).

The following abbreviations have been used for the different aminoacids in this paper.

Gly—Glycine	His—Histidine
Al—Alanine	Asp—Aspartic acid
Va—Valine	As—Asparagine
Ly—Lycine	Glu—Glutamic acid
Ar—Arginine	Leu—Leucine
Pro—Proline	Threo—Threonine

Analysis of wheat straw

Loss on ignition	90·5612	CaO	0·7562
Ash	9·4388	K_2O	0·6945
HCl insoluble	5·2897	MgO	0·1808
Sesquioxide	1·7423	Total carbon	36·6124
Fe_2O_3	0·9003	Total nitrogen	0·6024
P_2O_5	0·4869	C/N ratio	60·7

Analysis of the soil

Loss on ignition	3·9628	K_2O	1·0489
Ash	96·0372	MgO	1·0203
HCl insoluble	81·6210	Total carbon	0·4903
Sesquioxide	7·6210	Total nitrogen	0·0485
Fe_2O_3	3·9800	NH_3-N	0·0022
P_2O_5	0·1043	NO_3-N	0·0039
Available P_2O_5 (1% citric acid sol.)	0·0201	C/N ratio	10·1
CaO	1·4256		

Analysis of phosphates

%	Algerion rock phosphate	Trichinopoly rock phosphate	Bihar rock phosphate	Kulti basic slag	Tata basic slag	Belgian basic slag	German basic slag
SiO_2	56·68	16·31	25·52	31 89	33·64	33·26	22·75
Fe_2O_3	0 01	4·19	3·94	7·84	6·95	5·12	16·20
K_2O	9·0269	..	..	0·6100	0·907	1·032	0·9525
CaO	10·9593	20·80	11·7980	31·7741	30·3670	27·42	33·55
MgO	1·1582	0·92	0·32	3·0591	2·2875	2·00	5·505
Total P_2O_5	19·6345	27·5001	19·54	7·48	7·908	17·801	17·9050
Available P_2O_5	7·569	1·79	1·12	3·640	4·061	9·2390	9·3699
MnO	..	..	..	..	1·38	2·30	..

TABLE I

Average temperature 30°C

Period of exposure in days	Total N %	NH_3—N %	NO_3—N %	Total Available N%	Percent available N over Total N	Percent nitrate N over Total N	Aminoacids identified (Chromatographically)	Amount with respect to glycine in (mgm.) %
100 gms. of wheat straw + 10 gms. soil + 40% moisture								
LIGHT								
0	0·6072	0·0002	0·0004	0·0006	..	..	Gly,Al,Va.	..
50	0·7537	0·0221	0·0283	0·0504	6·6	56·1	Al,Va,Glu,Asp,Ly.	2·5250
90	0·8589	0·0362	0·0530	0·0892	9·3	59·4	Al,Va,Glu,Asp,Ly, Threo,Leu.	3·8615
120	0·9042	0·0464	0·0819	0·1283	13·8	63·9	Al,Va,Glu,Asp,Ly Threo,Leu.	4·5210
150	0·8781	0·0427	0·0779	0·1206	13·7	63·7	Va,Glu,Lv,Asp,Leu, Threo.	4·1142
DARK								
0	0·6072	0·0002	0·0004	0·0006	..	..	Gly,Al,Va.	..
50	0·6149	0·0141	0·0172	0·0313	5·0	54·6	Gly,Al,Va,Glu,Asp.	1·4551
90	0·7302	0·0253	0·0322	0·0575	7·8	56·0	Al,Va,Gly,Ly,Asp	2·5473
120	0·7578	0·0008	0·0478	0·0786	10·5	60·9	Al,Va,Glu,Ly,Asp, Threo.	2·9551
150	0·7454	0·0294	0·0461	0·0755	11·2	61·0	Va,Glu,Threo,Ly,Asp,Al.	2·7533

TABLE II

Average temperature 36°C

Period of exposure in days	Total N %	NH_3—N %	NO_3—N %	Total Available N %	Percent available N over Total N	Percent nitrate N over Total N	Aminoacids identified (Chromatographically)	Amount with respect to glycine in mgm. %
100 gms. of wheat straw + 10 gms. soil + 0·5 % P_2O_5 as German basic slag								
LIGHT								
0	0·6072	0·0002	0·0004	0·0006	. .	. .	Gly, Al, Va.	. .
50	0·8942	0·0294	0·0458	0·0752	8·4	60·2	Al, Va, Ly, Asp, Glu, As, Threo, Pro.	3·9268
90	1·1123	0·0511	0·0936	0·1447	13·4	64·7	As, Ly, Va, Asp, Glu, Ar, His, Threo, Leu.	5·8936
120	1·2187	0·0730	0·1669	0·2399	19·6	69·9	Va, Ly, Asp, Glu, As, Ar, His, Threo, Leu.	6·8264
150	1·1785	0·0674	0·1614	0·2288	19·2	70·0	Va, Asp, Glu, As, Ar, His, Threo, Leu.	6·2808
DARK								
0	0·6072	0·0002	0·0004	0·0006	. .	. .	Gly, Al, Va.	. .
50	0·7444	0·0193	0·0258	0·0451	6·0	57·2	Al, Va, Glu, Ly, Asp, Threo.	2·3820
90	0·8432	0·0343	0·0559	0·0902	10·6	61·9	Al, Va, Ly, Asp, Glu, Ar, His, Threo.	3·7943
120	0·9050	0·0478	0·0911	0·1389	15·3	65·5	Al, Va, Ly, Glu, Asp, Ar, His, Threo.	4·4345
150	0·8869	0·0453	0·0891	0·1344	15·1	66·2	Va, Ly, Glu, Asp, Ar, His, Threo.	4·1687

TABLE III

Average temperature 30°C

Period of exposure in days	Total N %	NH_3—N %	NO_3—N %	Total Available N%	Percent available N over Total N	Percent nitrate N over Total N	Aminoacids identified (Chromatographically)	Amount with respect to glycine in (mgm.) %
100 gms. of wheat straw + 10 gms. soil + 0·5% P_2O_5 as Trichinopoly rock phosphate								
LIGHT								
0	0·6072	0·0002	0·0004	0·0006	..	..	Gly,Al,Va.	..
50	0·8156	0·0245	0·0352	0·0597	7·3	58·7	Al,Va,Ly,Glu,Asp, Threo.	3·3439
90	0·9541	0·0426	0·0699	0·0125	11·7	63·2	Al,Va,Ly,Glv,Asp,Ar, Threo,His.	4·9713
120	1·0271	0·0562	0·1089	0·1651	16·0	65·9	Al,Va,Glu,Ly,Asp,Threo, His,Ar.	5·6490
150	0·9870	0·0516	0·1024	0·1540	15·6	66·4	Va,Glu,Ly,Asp,Threo, His,Ar.	5·1978
DARK								
0	0·6072	0 0002	0·0004	0·0006	..	..	Gly,Al,Va.	..
50	0·7017	0·0148	0·0193	0·0341	5·0	56·5	Gly,Al,Va,Ly,Glu,Asp.	2·0350
90	0·7755	0·0258	0·0398	0·0656	8·4	60·6	Al,Va,Glu,Asp,Ly,Threo.	3·2560
120	0·8147	0·0372	0·0642	0·1014	12·4	63·4	Al,Va,Glu,Ly,Asp,Threo, His.	3·5846
150	0·7953	0·0354	0·0621	0·0966	12·1	64·2	Va,Ly,Glu,Asp,Threo, His.	3 3698

TABLE IV

Average temperature 30°C

Period of exposure in days	Total N %	NH_3—N %	NO_3—N %	Total Available N%	Percent available N over Total N	Percent nitrate N over total N	Amino acids identified (Chromatographically)	Amount with respect to glycine in mgm. %
100 gms. wheat straw + 10 gms. soil + 0·5 % P_2O_5 as Tata basic slag								
LIGHT								
0	0 6072	0·0002	0·0004	0·0006	. .	. .	Gly,Al,Va.	. .
50	0 8852	0·0275	0·0438	0·0698	7·8	60·4	Al,Va,Ly,Glu,Asp,Threo, As	3·8048
90	1·0964	0 0494	0·0892	0·1386	12·6	64·3	Va,Ly,Asp,Glu,As,His, Threo,Leu.	5·6960
120	1·1982	0 0659	0·1489	0·2148	18·3	69·3	Ly,Va,Asp,Glu,Threo, As,His,Ar,Leu.	6·5901
150	1·1583	0 0611	0 1400	0·2041	17·6	70·0	Va,Asp,Glu,As,Threo, His,Ar,Leu.	6·0629
DARK								
0	0·6072	0·0002	0·0004	0·0006	. .	. .	Gly,Al,Va.	. .
50	0·7372	0·0176	0 0232	0·0408	5·5	56 8	Al,Va,Ly,Glu,Asp,Threo.	2·2116
90	0·8303	0 0320	0·0494	0 0814	9·8	60·6	Al,Va,Glu,Asp,As,Threo, His	3·6533
120	0·8885	0·0449	0·0815	0·1264	14·2	64·4	Al,Va,Ly,Glu,Asp,His, Threo,Ar.	4·1759
150	0 8694	0·0424	0·0794	0·1218	14 0	65·1	Va,Ly,Glu,Asp,Threo, His,Ar.	3·9257

TABLE V

Average temperature 30°C

Period of exposure in days	Total N %	NH_3—N %	NO_3—N %	Total Available N%	Percent available N over Total N	Percent nitrate N over total N	Aminoacids identified (Chromatographically)	Amount with respect to glycine in mgm. %
100 gms. of wheat straw + 10 gms. soil + 0·5% P_2O_5 as Kluti basic slag								
LIGHT								
0	0·6072	0·0002	0·0004	0·0006	..	..	Gly,Al,Va.	..
50	0·8480	0 0265	0·0392	0·0657	7·7	59·6	Al.Va,Ly,Asp,Glu,As, Threo.	3·5616
90	1·0294	0·0453	0·0809	0·1262	12·2	64·1	Al,Va,Ly,As,Asp,Glu,Ar, Threo.	5·3499
120	1·1080	0·0592	0·1329	0·1921	17·3	69·1	Va,Ly,Glu,Asp,As,His, Threo,Ar,Pro.	6·0832
150	1·0656	0·0554	0·1254	0·1808	16·9	69·3	Va,Glu,Asp,As,His,Threo, Pro,Ar.	5·5968
DARK								
0	0·6072	0·0002	0·0004	0·0006	..	..	Gly,Al,Va.	..
50	0·7223	0·0173	0·0213	0·0386	5·3	55·1	Al,Va,Ly,Glu,Asp,Threo.	2·0946
90	0·8091	0·0308	0·0445	0·0753	9·4	59·0	Al,Va,Glu,Asp,Ly,His, Threo,Pro.	3·3982
120	0·8526	0·0417	0·0715	0·1132	13·2	63·1	Va,Ly,Glu,Pro,His,Asp, As,Threo.	3·8367
150	0·8329	0·0392	0·0704	0·1096	13·1	64·2	Va,Ly,Glu,Pro,Threo, His,Asp,As.	3·6069

TABLE VI

Average temperature 30°C

Period of exposure in days	Total N %	NH_3—N %	NO_3—N %	Total Available N%	Percent available N over Total N	Percent nitrate N over total N	Amino acids identified (Chromatographically)	Amount with respect to glycine in mgm. %
100 gms. of wheat straw + 10 gms. soil + 0·5% P_2O_5 as Algerian rock phosphate								
LIGHT								
0	0·6072	0·0002	0·0004	0·0006	. .	. .	Gly,Al,Va.	. .
50	0·8244	0·0256	0·0364	0·0620	7·5	58·7	Al,Va,Ly,Glu,Asp, Threo.	3·4624
90	0·9811	0·0424	0·0732	0·1156	11·7	63·31	Al,Va,Ly,Glu,Asp,His, Threo,Ar.	5·1017
120	1·0588	0·0591	0·1149	0·1740	16·4	66·0	Al,Va,Ly,Glu,Asp,His, Threo,Ar.	5·8234
150	1·0261	0·0582	0·1142	0·1724	16·7	66·2	Va,Ly,Glu,Asp,Threo, His,Ar.	5·3576
DARK								
0	0·6072	0·0002	0·0004	0·0006	. .	. .	Gly,Al,Va.	. .
50	0·7073	0·0170	0·0197	0·0367	5·2	53·7	Al,Va,Ly,Glu,Asp.	2·0511
90	0·7851	0·0285	0·0423	0·0708	9·0	59·7	Al,Va,Ly,Glu,Asp,Threo, His.	3·3189
120	0·8311	0·0382	0·0675	0·1057	12·7	63·9	Al,Va,Ly,Glu,Asp,Threo, His.	3·7268
150	0·8100	0·0355	0·0652	0·1007	12·4	64·7	Va,Glu,Ly,Asp,Threo, His.	3·5036

TABLE VII

Average temperature 30°C

Period of exposure in days	Total N %	NH_3—N %	NO_3—N %	Total Available N%	Percent available N over Total N	Percent nitrate N over Total N	Aminoacids identified (Chromatographically)	Amount with respect to glycine in mgm. %
100 gms. wheat straw + 10 gms. soil + 0·5 % P_2O_5 as Belgian basic slag								
LIGHT								
0	0·6072	0·0002	0·0004	0·0006	..	..	Gly,Al,Va.	..
50	0·8865	0·0301	0·0429	0·0729	8·2	58·7	Al,Va,Asp,Glu,As, Threo,Ly.	3·9006
90	1·0973	0·0528	0·0899	0·1427	13·2	63·1	Va,Ly,Asp,Glu,As,Ar, His,Threo.	5·8156
120	1·2019	0·0753	0·1463	0·2216	18·4	66·0	Leu,Va,Ly,Asp,Glu,As, Ar,His,Threo.	6·7306
150	1·1600	0·0694	0·1405	0·2099	18·0	66·9	Va,Asp,Glu,As,Ar,His, Threo,Leu.	6·1922
DARK								
0	0·6072	0·0002	0·0004	0·0006	..	..	Gly,Al,Va.	..
50	0·7406	0·0196	0·0247	0·0443	5·9	55·7	Al,Va,Ly,Glu,Asp.	2·3699
90	0·8362	0·0356	0·0519	0·0875	10·4	59·3	Al,Va,Glu,Ly,Asp, His,Threo.	3·7629
120	0·8969	0·0485	0·0841	0·1326	14·7	63·4	Va,Ly,Glu,Asp,His,Al, Threo,Ar.	4·3051
150	0 8783	0·0460	0·0820	0·1280	14·5	64·0	Va,Ly,Glu,Asp,Threo, His,Ar.	4·0471

DISCUSSION

The results recorded in the foregoing tables show that during the composting process there is a gradual increase in the available nitrogen content of the system and that increase is more pronounced in light than in the dark. Moreover, in presence of phosphates the amount of available nitrogen released is further increased. This is probably due to the alkaline nature of the phosphates and alkalinity favours oxidation is a well known fact. It has also been observed that amount of available nitrogen is invariably higher in the system containing basic salg than rock phosphates. This again is due to more alkaline nature of basic slags.

The reasons for considerably higher percentage of available nitrogen in the phosphated systems is not far to seek. Under ordinary conditions organic matter on decomposition and oxidation undergoes the following reactions.

$$\text{Organic matter} \rightarrow \text{Proteins} \rightarrow \text{Aminoacids} \xrightarrow{+O_2} NH_3 \xrightarrow{+O_2} NO_2 \xrightarrow{+O_2} NO_3$$

In this series of reactions highly unstable compound *i.e.* ammonium nitrite is formed which decomposes releasing energy and nitrogen gas and water as in the following equation.

$$NH_4\,NO_3 \rightarrow N_2 + 2H_2O + 718 \text{ K Cal.}$$

This reaction is catalysed by acidity. The alkaline phosphate when added to the decomposing organic matter reduces the decomposition of ammonium nitrite by decreasing the acidity and supplying calcium, Magnesium and Potassium ions to form Calcium, Magnesium and Potassium nitrites which are more stable than ammonium nitrite, and thus increases the concentration of available nitrogen.

The effectiveness of different phosphates in producing available nitrogen is in the following order :

German basic slag > Belgium basic salg > Tata basic slag > Kulti basic slag > Algerian rock phosphate > Trichinopoly rock phosphate > Control.

A perusal of the foregoing results shows that rate of nitrification as determined by the nitrate nitrogen percent over the total nitrogen is always greater in composts exposed to light than in those kept in the dark. This seems to be due to the phtochemical oxidation of the ammonium salts to nitrate. The nitrification is also accelerated by the incorporation of phosphates with decomposing organic matter. Recently Dhar *et al*, (1961) reported that in presence of light and inductors, nitrite solutions can be oxidised to nitrate under completely sterile condition and that oxidation is appreciably increased in presence of phosphates.

The stimulative effect of different phosphates on the process of nitrification is in the following order :

German basic siag > Tata basic slag > Kulti basic slag > Belgian basic slag > Algerian rock phosphate > Trichinopoly rock phosphate > Control.

It is interesting to record that the number and amount of free aminoacids was found to be greater in the phosphated and exposed sets than in the unphosphated and covered ones. Rothwell and Frederic (1956) observed aminoacids during the decomposition of alfalfa and corn stover in soil at different temperatures.

Recently Dhar and Roy (1955) have obtained aminoacids by exposing normal solutions of nitrates and carbonydrates with titania as photosensitizer. They have also shown that in presence of phosphates formation of aminoacids is enhanced. In soils also, the presence of nitrate along with carbohydrate is well known. It is just likely that in presence of light, this nitrate nitrogen may get reduced to amino nitrogen and in this form may join the carbohydrate chain and thus form aminoacids. In the phosphated system phosphates may react with these aminoacids and form more or less stable compounds (Dhar and Ghosh 1956) and thus make these resistant towards decomposition and oxidation. It seems, therefore, that light helps in the synthesis of aminoacids and phosphates increase their concentration by making these resistant towards oxidation and decomposition.

From these observations it can be concluded that the release of available nitrogen and free aminoacids from the organic matter undergoing oxidation and decomposition is greatly increased by the action of light and phosphates. Thus the manurial value of the composts can be considerably increased by the joint action of light and phosphates.

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UNIFORM RADIAL OSCILLATIONS OF A ROTATING MAGNETIC STAR

By

A. C. BANERJI* *and* S. K. GURTU**

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ABSTRACT

The condition for the uniform radial oscillations of a rotating magnetic star is determined. *Prima facie* it appears that uniform radial oscillations are not possible. However, under stringent conditions the star may execute such mode of oscillations.

Sterne[1] has suggested that the probability of stability of radial oscillations is greatest for the homogeneous non-magnetic star and even for this the fundamental mode would be the prominent one.

Sen[2] has shown that amongst the non rotating stars only the homogeneous sphere is capable of uniform oscillation. His expression for the frequency of oscillation is same as Sternes.[1] In the same paper he has further shown that superposition of a small amount of rotation does not affect the period of uniform oscillation of a homogeneous spherical star. In another paper[3] he has demonstrated that fast rotating stars can not execute radial mode of uniform oscillation, his analysis supports the observational fact of the non existence of fast rotating Cepheids.

Bhatnagar[4] considered radial oscillations of a rotating star. He has obtained an expression for the frequency of oscillation, from which he has concluded that rotation decreases the period of uniform radial oscillation of the model if $\gamma < \frac{5}{3}$, where γ is the ratio of the specific heats, this result had been obtained independently by Ledoux[5] too.

The former author has elsewhere[6] considered the quasi-uniform radial osillations of magnetic stars. From which it appears that generally quasi-uniform oscillations are not possible.

Since actually stars do possess rotation, hence in the present paper rotation has been considered as well. Following Bhatnagar[4] we have considered the stellar model in which strata of equal density are similar spheroids and which is uniformly rotating so that at any given instant the angular velocity is same throughout the mass.

Let p, f, g, and ω be the pressure, density, gravity, and angular velocity at a point distant r from the centre at any instant t at a point (r, θ, ϕ) and let the suffix denote the undisturbed values of the variables.

Let $r = r_0 (1 + r_1)$; $p = p_0 (1 + p_1)$; $\rho = \rho_0 (1 + \rho_1)$ and $w = w_0 (1 + w_1)$ (1)

If the oscillations are adiabatic[7]

$$p_1 = \gamma \rho_1 \tag{2}$$

where γ is the ratio of the specific heats.

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*Ex-Vice-Chancellor and Emeritus Professor, Allahabad University, 4-A, Beli Road, Allahabad.

**Research Scholar, Department of Mathematics, Allahabad University, Allahabad.

Let a_0, a_0 and $a_0\,(1-\varepsilon)$ be the semi axes of the spheroid through the point (r_0, θ, ϕ), where ε is the ellipticity of the spheroid, and these expand to a, a and $a(1-\varepsilon)$. The volume of the thin spheroidal shell of semi axes a, a and $a(1-\varepsilon)$ is $4\pi\lambda\, a^3\,(1-\varepsilon)$

where
$$\lambda = \frac{da}{a} = \frac{dr}{r}$$

$$\rho_0\, a_0^3\,(1-\varepsilon)\,\frac{dr_0}{r_0} = \rho\, a^3\,(1-\varepsilon)\frac{dr}{r} \tag{3}$$

It can be seen that a and $a(1-\varepsilon)$ are the semi major and semi minor axes of the meridian saction of the spheroid. Neglecting the square of the ellipticity the polar equation[8] of the surface is

$$r = a\,(1-\varepsilon\,\cos^2\theta) \tag{4}$$

The approximation being valid as according to Clairaut's theorem on rotating bodies

$$\varepsilon \;\alpha\; \frac{\omega^2}{2\pi\, G\, \bar{\rho}}$$

where $\bar{\rho}$ is the mean density and G is the gravitational constant.

From (3) and (4)

$$\rho_0\, r_0^2\, dr_0 = \rho\, r^2\, dr \tag{5}$$

From (1)

$$\frac{dr}{dr_0} = 1 + r_1 \tag{6}$$

since for uniform osicllation γ_1 is indcpendent of r_0

From (1), (5), and (6)

$$1 + \rho_1 = \frac{\rho}{\rho_0} = 1 - 3\; r_1 \tag{7}$$

to the first powers of γ_1

$$\text{or } \rho_1 = -3\, r_1 \tag{8}$$

From (2) and (8

$$p_1 = -3\,\gamma r_1 \tag{9}$$

The stellar magnetic field is assumed to be derived from a scalar polential of the form $\frac{S_n}{r^{\,n+1}}$, where S_n is the surface harmonics of degree indicated by the subscript, a simple possible from for the magnetic field[9] is

$$H = \frac{\rho}{r^{n+2}} \tag{10}$$

since uniform radial oscillations in a fixed direction of θ and ϕ are being considered hence l is taken as constant, n in the above equation is a positive integer.

Since the star is in rotation, any field will just rotate without suffering any distortion. Therefore taking the magnetic field to be constant along a line of

force, and to vary from one line of force to another it can be seen that the affect of the magnetic field is to increase the pressure by $\frac{\mu H^2}{8\pi}$, where μ is the magnetic permeability, thus the total pressure will be $p + \frac{\mu H^2}{8\pi}$.

Electric forces can be neglected because of infinite electrical conductivity assumed for the stellar material.

Taking into account the magnetic field, the rotation of the star about the z- axis, and neglecting viscous forces in the general hydrodynamical equation, the polar form of equation of motion is.

$$\frac{1}{\rho}\frac{d}{dr}\left(p + \frac{\mu H^2}{8\pi}\right) = \frac{\partial V}{\partial r} - \frac{d^2 r}{dt^2} + w^2 r \sin^2\theta \tag{12}$$

where V is the gravitational potential at a point (r, θ, ϕ), the term $w^2 r \sin^2\theta$ is due to rotation.

From (10) and (12)

$$\frac{1}{\rho}\frac{dp}{dr} = \frac{\mu_1^2}{8\pi\rho}\frac{2n+4}{r^{2n+5}} + \frac{\partial V}{\partial r} - \frac{d^2 r}{dt^2} + w^2 r \sin^2\theta \tag{13}$$

Now (1) and (5) give

$$\frac{1}{\rho}\frac{dp}{dr} = \frac{1}{\rho}\frac{dp}{dr_0}\frac{dr_0}{dr} = \frac{1}{\rho_0}\frac{dp_0}{dr_0}(1 + r_1)^2 \tag{14}$$

From (13) and (14)

$$\frac{1}{\rho_0}\frac{dp}{dr_0}(1 + r_1)^2 = -g + \sigma^2 r_0 r_1 + w^2 r \sin^2\theta + \frac{\mu_1^2}{8\pi\rho}\frac{2n+4}{r^{2n+5}} \tag{15}$$

where $\frac{2\pi}{\sigma}$ gives the period of pulsation

The value of gravity at a point (r, θ, ϕ) is

$$g = -\frac{\partial V}{\partial r} = \frac{G M(r)}{r^2} + \frac{3a^3}{r^4}\left(\tfrac{1}{2} w^2 a^2 - \frac{G M(r)\varepsilon}{a}\right)(\cos^2\theta - \tfrac{1}{3}) \tag{16}$$

where $M(r)$ is the mass of the spheriod interior to (r, θ, ϕ)

From (4) and (16)

$$g = \frac{G M(r)}{a^2} + \frac{3\cos^2\theta - 1}{2} w^2 a + \frac{G M(r)}{a^2}\varepsilon \sin^2\theta$$

$$= \frac{G M(r)}{a^2}\left[1 + \varepsilon \sin^2\theta + \frac{3\cos^2\theta - 1}{2C M(r)}\omega^2 a^3\right] \tag{17}$$

and

$$g_0 = \frac{G M(r_0)}{a_0^2}\left[1 + \varepsilon \sin^2\theta + \frac{3\cos^2\theta - 1}{2G M(r_0)}\omega_0^2 a_0^3\right] \tag{18}$$

where $M(r) = M(r_0)$ due to conservation of mass.

From (17) and (18)

$$\frac{g}{g_0} = 1 - r_1 \left[2 + \frac{w_0^2 \, r_0^3}{2G \, M(r_0)} \; (3 \cos^2\theta - 1) \right] \tag{19}$$

Using (1), (15) and (19)

$$\frac{1}{\rho_0} \frac{dp}{dr_0} = - \frac{1}{(1 + r_1)^2} \, g_0 \left[1 - r_1 \left\{ 2 + \frac{w_0^2 r_0^3}{2G \, M(r_0)} \; (3 \cos^2\theta - 1) \right\} \right]$$

$$+ \frac{\sigma^2 r_0 r_1}{(1+r_1)^2} + \frac{w_0^2 \, (1+w_1)^2}{(1+r_1)^2} \, r_0 (1+r_1) \sin^2 \theta + \frac{\mu_1^2}{8\pi\rho_0 \, (1+\rho_1)} \; \frac{2n+4}{r_0^{2n+5} \, (1+r_1)^{2n+7}} \tag{20}$$

Using (1), (8), (20), and the relation $\omega_1 = - 2r_1$

$$\frac{1}{\rho_0} \cdot \frac{d}{dr_0} \; (p_0 + p_0 \, p_1) = - g_0 + \left[4g_0 + \frac{g_0 w_0^2 r_0^3 \, (3 \cos^2\theta - 1)}{2G \, M(r_0)} \right] r_1 + \sigma^2 \, r_0 r_1$$

$$+ \omega_0^2 \, (1 - 5r_1) \, r_0 \sin^2 \theta + \frac{\mu_1^2 \, (2n+4)}{8\pi\rho_0 \, r_0^{2n+5}} \left[1 - (2n+4) \, r_1 \right]$$

which breaks up into the equation of magnetohydrostatic equilibrium in the undisturbed state as

$$\frac{1}{\rho_0} \frac{dp_0}{dr_0} = - g_0 + \frac{\mu_1^2}{8\pi\rho_0} \; \frac{(2n+4)}{r_0^{\;2n+5}} + w_0^2 \; r_0 \sin^2\theta \tag{21}$$

and

$$\frac{1}{\rho_0} \frac{d}{dr_0} \, (p_0 p_1) = g_0 \left[4 + \frac{w_0^2 \, r_0^3 \; (3 \cos^2\theta - 1)}{2G \; M(r_0)} \right] r_1 + \sigma^2 \, r_0 \, r_1$$

$$- 5 \; w_0^2 r_0 r_1 \sin^2 \theta - \frac{\mu_1^2 \; (2n+4) \; (2n+4)}{8\pi\rho_0 \; r_0^{2n+5}} \; r_1 \tag{22}$$

Using (9), (21), and (22)

$$- 3\gamma r_1 \left[- g_0 + \frac{\mu_1^2}{8\pi\rho_0} \cdot \; \frac{(2n+4)}{r_0^{2n+5}} + w_0^{\;2} r_0 \; \sin^2\theta \; \right]$$

$$= g_0 \; \left\{ \; 4 + \frac{w_0^2 r_0^5}{2GM(r_0)} \, (3 \cos^2\theta - 1) \right\} r_1 \; + \sigma^2 r_0 r_1 - 5 \; w_0^2 r_0 r_1 \sin^2\theta - \frac{\mu_1^2 \, (2n+4)^2 r_1}{8\pi\rho_0 \; r_0^{2n+5}}$$

Hence

$$\sigma^2 \; = \; \frac{g_0}{r_0} \left[(3\gamma - 4) - w_0^2 \; r_0^3 \, \frac{(3 \cos^2\theta - 1)}{2G \; M(r_0)} \right]$$

$$+ \, (5 - 3\gamma) \; w_0^2 \, \sin^2\theta \, + \, \frac{\mu_1^2}{8\pi\rho_0 \; r_0^{2n+6}} \, (2n+4) \, (2n+4 - 3\gamma) \tag{23}$$

It is interesting to note that if affect due to magnetic field is neglected then the expression for σ^2, as proposed by Bhatnagar,[4] for a rotating non-magnetic star, will be obtained as

$$\sigma^2 = \frac{g_0}{r_0}\left[(3\gamma - 4) - w_0^2 r_0^3 \frac{(3\cos^2\theta - 1)}{2GM(r_0)} \right] + (5 - 3\gamma)\, w_0^2 \sin^2\theta \tag{24}$$

Also it can be seen that if rotation terms are neglected then the expression for σ^2, as proposed by the former author elsewhere[6], for a non-rotating magnetic star, will be

$$\sigma^2 = \frac{g_0}{r_0}\cdot(3\gamma - 4) + \frac{\mu\rho^2}{8\pi\rho_0\, r_0^{2n+6}}\,(2n+4)\,(2n+4-3\gamma) \tag{25}$$

The condition for the uniform oscillation to be finite is

$$\frac{g_0}{r_0}\left[(3\gamma \quad 4) - w_0^2 r_0^3 \frac{(3\cos^2\theta - 1)}{2G\,M(r_0)}\right] + (5 - 3\gamma)\, w_0^2 \sin^2\theta$$

$$+ \frac{\mu_1^2\,(n+2)}{4\pi\rho_0\, r_0^{\;2n+6}}\,(2n+4-3\gamma) > 0 \tag{26}$$

and that it should be independent of r_0. It appears *prima facie* that uniform oscillations are not possible. Nevertheless, under stringent conditions it can be seen that uniform radial oscillations are possible in case of a slowly rotating inviscid magnetic star.

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ELECTROMETRIC TITRATIONS OF BENTONITE WITH DIFFERENT EXCHANGE CATIONS IN AQUEOUS AND NON-AQUEOUS MEDIUM

By

WAHID U. MALIK *and* G. C. GUPTA

Chemical Laboratories, University of Roorkee, Roorkee

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ABSTRACT

Potentiometric and conductimetric titrations of hydrogen, calcium, magnesium, and iron bentonite have been carried out both in the aqueous and non-aqueous media using caustic soda and calcium hydroxide as the titrants. With the different exchange forms of the clays two inflexion points are obtained both in the aqueous and non-aqueous medium but with the original clay or with the mixtures of different exchange forms of the clay the number of inflexion points is increased from four to five and three to five respectively, in passing from the aqueous to the non-aqueous media. The pK values as obtained from the titration curves were found to fall under two categories (*i*) calcium and magnesium clays (*ii*) hydrogen and iron clays giving almost similar pK values. This similarity of behaviour of ferric and hydrogon clay has been explained on the basis of the hydrolysis of the highly hydrated ferric ions of the outer double layer of the clay giving out hydrogen ions. The titrations with $Ca(OH)_2$ gave different pK values from those realized in the case of caustic soda. The cation exchange capacities (c. e. c.) were different for the first inflexion point in the case of two alkalies but were almost the same for the second inflexion point. The c. e. c. values of the clays were in the order.

$$H^{+} > Fe^{+++} > Mg^{++} > Ca^{++}$$

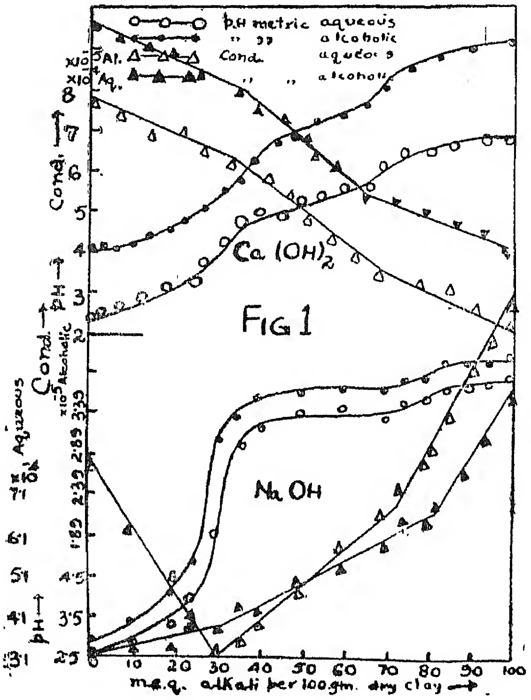
The increase in the number of inflexion points in non-aqueous medium is presumed due to some change taking place in the alumina structure of the clay in the presence of non-electrolytes.

The use of electrometric titrations as a means of identifying clay minerals is well recognized. The technique has been extensively employed by Paver and Marshall[1], Harman and Fraulin[2], Mukherjee and Mitra[3], Grim[4], Adhikari[5] and Chakrawarti[6] in the case of hydrogen clays or H–Al clays (as designated by Harward and Coleman[7]) but the technique has so for not been extended to clays having exchange cations other than hydrogen. It was therefore thought desirable to work on these lines choosing such exchage clays, as those important either from the technological or agricultural view point. In the present communication results of potentiometric and conductometric titrations in aqueous and non aqueous medium are described with Calcium, Magnesium, Iron and Hydrogen as the exchange cations on a bentonite clay sample. The non aqueous studies were particularly undertaken in order to account either the levelling effect of the aqueous medium (Ruehle[8], Fritz[9], Palit[10], Singhal and Malik[11]) or to assess the significance of the so called similarity of clays to weak soluble acids (Puri[12]).

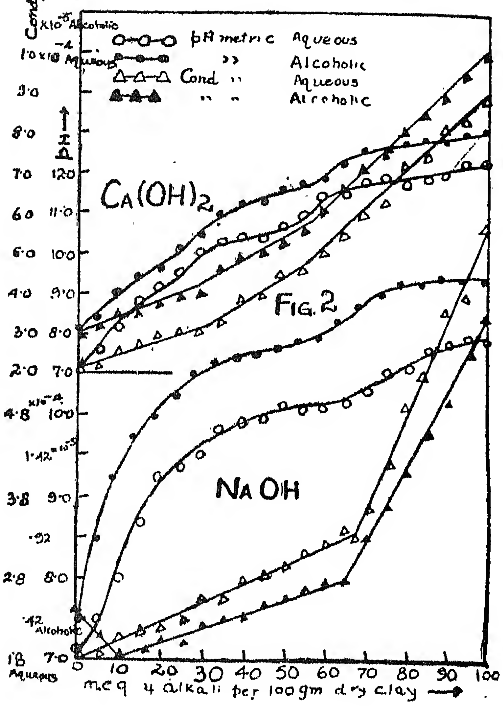
METHODS AND MATERIALS

The materials were used from BDH analytical reagents. The organic liquids were purified by double distillation. Doubly distilled (all glass) water was used. All the containers were made of Pyrex glass. B. D. H. technical Bentonite tested for the absence of organic matter and other water soluble impurities was used throughout the work.

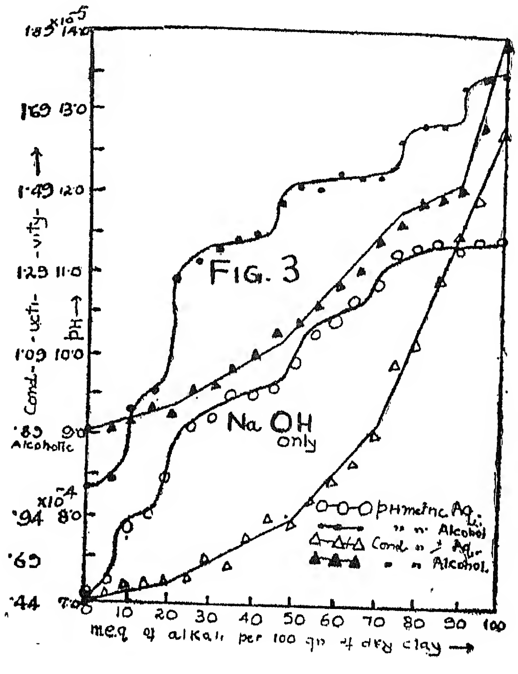
Hydrogen Bentonite
Titration Curves



Calcium Bentonite
Titration Curves



Original Bentonite
Titration Curves



PREPARATION OF CLAY SAMPLES

A known weight of the clay was added to a measured quantity of double distilled water and was kept shaking for more than two hours in a mechanical shaker. It was then centrifuged for half an hour at 3000 r.p.m. and the colloidal solution so obtained was used for the preparation of different exchange-clay samples. The H^+, Ca^{++}, Mg^{++}, Fe^{+++}, sols were obtained by the ion exchange method as recommended by Harward and Coleman[7], and Slabough[13]. Amberlite IR 120 columns ($30'' \times 1''$) were treated with IM Hydrochloric acid and 0·1M Calcium acetate, Magnesium acetate, and ferric citrate. Almost neutral salts were taken to avoid the formation of a mixed acidic clay. In the case of Calcium and Magnesium columns the rate of flow had to be sometimes increased by gentle suction. The sols were passed a number of times through the respective regenerated columns till on repeated treatment no change in the pH of the sol was observed.

PROCEDURE

Ten ml. of the sol were added to about fifteen boiling tubes each. Varying amounts of N/50 NaOH in aqueous or nonaqueous medium were added and the total volume was made up to 20 ml. The tubes were then well corked and kept for about sixty hours to attain equilibrium.

Before carrying out the actual measurements preliminary experiments were first conducted to determine the approximate inflexion points so that a large number of readings could be taken just before and after the equivalence point. In the case of clay sols containing two exchange cations a ratio of 1 : 1 was maintained while for those having three cations the ratio was 6 Mg : 5 Ca : 3 Fe. Titrations were also done with $Ca(OH)_2$ (in 50% nonaqueous medium). The solvents were also titrated against both NaOH and $Ca(OH)_2$ in order to ensure accuracy of results in all the mediums.

APPARATUS

A Cambridge Bench Type pH meter was used for pH measurements. The readings obtained with this glass calomel assembly were checked with the Pye Students Potentiometer using a hydrogen electrode. Conductivity measurements were carried out with the help of Philips Conductivity Bridge Type PR 9500/90. All measurements were carried out at $30 \pm 0.05°C$.

Readings in the case of Hydrogen electrode were slightly lower (more so in acetone, methyl alcohol and ethyl alcohol) than the glass electrode. This however, did not interfere with the shape of the curve and the position of the inflexion points. pK values were determined by the relationship

$$pH = pK + \log \frac{\text{salt}}{\text{acid}}$$

pK_1, pK_2, pK_3 .... etc. were determined from the half neutralization points determined from the first, or second, or third inflexion points respectively.

DISCUSSION

The hydrogen bentonite as well as the corresponding Calcium, Magnesium and Iron clays give two inflexion points in the case of both the potentiometric and conductometric titrations. There is nothing significant to report as far as the difference between the number of inflexion points in the aqueous and non aqueous media are concerned except for the fact that the pK values are different in the two media (Table 1 and 2).

TABLE 1*

Base Exchange Capacities[1] and pK[2] values as calculated from pH metric and conductimetric titrations.

*Results are given only for two typical mediums. Almost similar results were obtained with glycerine, acetone and methyl alcohol as the medium.

1. B. e. c. is given in m. e. q. per 100 gms of the dry clay.
2. pK values at the respective inflexion points are given below the b. e. c. values.

Cation on the Clay	Method	Aqueous				Alcoholic			
		NaOH		$Ca(OH)_2$		NaOH		$Ca(OH)_2$	
		pK_1	pK_2	pK_1	pK_2	pK_1	pK_2	pK_1	pK_2
	pH	30	75	35	75	30	75	35	65
Hydrogen		3·8	7·9	2·9	4·0	4·3	9 0	4·8	5·6
	Cond.	30	70	35	65	30	80	35	65
	pH	25	75	32	65	26	75	35	65
Ferric		3·6	8·1	4·9	5·3	4·1	9·4	6·2	7·0
	Cond.	25	75	30	65	25	75	30	65
	pH	12	65	27	55	12	65	28	60
Calcium		7·5	9·8	8·0	9·1	8 9	10·7	8·1	9·5
	Cond.	–	70	30	55	10	65	30	60
	pH	20	75	25	64	20	75	25	65
Magnesium		7·3	9·9	8·4	10·0	7·5	10·3	9·1	10 8
	Cond.	20	75	25	65	20	75	25	65

The order of base exchange capacities (b. e. c.) as determined from the inflexion points of the pH titration curves and the breaks in the conductimetric titration curves is of the order

$$H^+ > Fe^{+++} > Mg^{++} > Ca^{++}$$

The concordance between the values obtained by the two methods used is worth noticing.

Other interesting features of these titrations are :

(*i*) the b. e. c. values were higher in the case of calcium hydroxide as the titrant.

(*ii*) the nearness in the pK values of hydrogen and iron clay on the one hand and calcium and magnesium clays on the other hand.

(*iii*) existence of two breaks instead of one for hydrogen bentonite.

As regards (*iii*) it appears that the bentonite sample used is not pure but contains either illite (b. e. c. 10–40) or chlorite (b. e. c. 20-40) since the value of b. e. c. obtained at the first inflexion point is nearly 30 m. e. q. per 100 gms. of the dry clays (Fig. 1). Moreover, since the valves of b. e. c. for the first inflexion point differ for the four clays used it may be inferred that the chlorite or the illite content of the mineral is easily exchangeable with Ca^{++}, Mg^{++}, or Fe^{+++} ions.

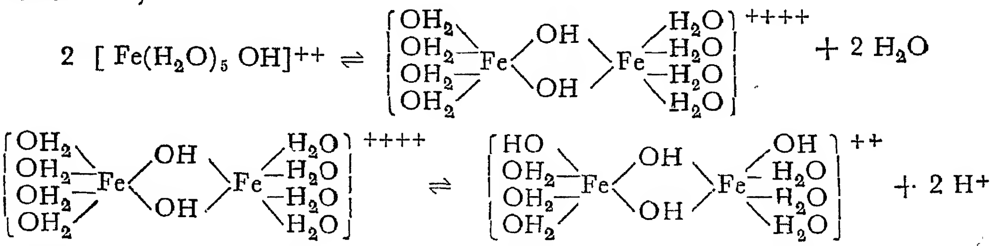
Although the order of b. e. c. ($H^+ > Fe^{+++} > Mg^{++} > Ca^{++}$) follows the well established behaviour *viz.* higher the potential of the clay mineral greater the ion exchange capacity. It is difficult to explain the difference in the exchange values obtained with NaOH and $Ca(OH)_2$ as the titrants, simply on this basis. No doubt that the higher valency of Ca^{++} ions is able to decrease the potential to a greater value than Na^+ ions, one cannot byepass the phenomenon of exchange adsorption on the addition of the gradually increasing amount of the titrant.

Values almost twice those obtained with NaOH (in the case of Ca clay against $Ca(OH)_2$. (Fig. 2) point towards the fact that in case of calcium exchange adsorption takes place so strongly that a new outer layer strongly attracted by the inner layer is obtained so that the potential drops very sharply resulting in the release of a large number of cations belonging to the outer part of the original double layer.

The close similarity in the electrometric behaviour of iron and hydrogen clays may be explained by assuming that the ferric ions of the outer layer are highly hydrated and undergo hydrolysis as given below :

$$Fe(H_2O)_6^{+++} \rightleftharpoons [Fe(H_2O)_5(OH)]^{++} + H^+$$

followed by



rendering the clay highly acidic approximating almost to a hydrogen clay.

The calcium and magnesium clays should not normally behave as acid clays since a hydrolytic mechanism as given above is not likely to exist here. Therefore the only way by which the acidic behaviour can be explained is that both the clays are not purely calcium or magnesium clays, but are Ca–H or Mg-H clays and that the magnitude of their acidity would depend upon the extent to which the hydrogen ions have been exchanged by the respective cations. In fact it is not possible to designate any clay as purely hydrogen or calcium or magnesium or iron clay. These are all mixed hydrogen clays and the pK values are the average for such mixed systems.

The change from the aqueous to the nonaqueous medium although does not influence the dissociation of the mono base exchange clays it does effect the original clay as well as the mixtures of exchange clays (Table 2). Thus the pH-metric curves show four inflexion points in the aqueous medium whereas five inflexion points are obtained in the non-aqueous media (Fig. 3). The conductivity data also gives an extra break in the non-aqueous media although one number less than in the pH curves. These results cannot be explained in terms of the suppression in the dissociation constants in presence of non electrolytes (the pK values not following the order of the dielectric constant) nor in terms of the decrease in hydrolytic tendency. It appears that the presence of the non electrolytes helps in changing the alumina structure of the clay or that of the ferric part of the mixed clays in such a manner as to make available new acidic sites for reaction with the alkali.

TABLE 2

Bentonite Clay containing more than one cation against NaOH only

Base Exchange Capacities and pK values as calculated in Table 1.

Cations on the Clay	Method	Aqueous pK_1	pK_2	pK_3	pK_4	pK_5	Alcoholic pK_1	pK_2	pK_3	pK_4	pK_5
Hydrogen	pH	15	40	60	–	85	10	40	55	75	90
+		3·5	4·5	4·7	–	5·3	4·4	6·0	6·4	7·2	7·8
Ferric	Cond	15	40	60	–	–	10	40	55	75	–
Calcium	pH	–	30	40	75	–	–	30	45	70	85
+		–	7·4	7·8	9·3	–	–	8·4	8·7	10·5	10·7
Magnesium	Cond	–	–	45	75	–	–	30	45	70	80
6 Magnesium	pH	8	20	48	75	–	10	25	45	65	90
+ 5 Calcium		6·4	7·0	7·9	8·2	–	6·2	7·3	7·5	8·3	9·0
+ 3 Ferric	Cond	–	20	50	75	–	–	25	45	65	85
Original Clay	pH	8	20	50	72	–	10	20	45	75	90
$(Ca^{++}+Mg^{++}$		7·2	8·0	9·1	9·5	–	8·5	9·1	11·1	11·4	11·7
$+ Fe^{+++}+Na^{+})$	Cond	–	20	50	70	–	–	20	45	75	90

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HEAT CONDUCTION AND HERMITE POLYNOMIALS

By

B. R. BHONSLE

Department of Applied Mathematics, Government Engineering College, Jabalpur

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ABSTRACT

The object of this note is to make use of Hermite polynomials in solving the partial differential equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - k\,u\,x^2$$

when $u(x, t)$ tends to zero for large values of t and when $|x| \to \infty$. The equation is related to a problem of heat conduction.

Hermite polynomials have been utilised by Kampé De Fériet[2] in solving a heat conduction equation. He has obtained four theorems which are of the nature of existence theorems.

The Hermite polynomials as defined by Sneddon [4, p, 150] is slightly different. Sneddon has defined the Hermite polynomials $H_n(x)$ for integral values of n and all real values of x by the identity,

$$e^{2xt-t^2} = \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n \tag{1·1}$$

We shall adopt the above definition of $H_n(x)$.

The object of this note is to make use of the Hermite polynomials in solving the following differential equation,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - k\,u\,x^2 \tag{1·2}$$

under some initial and boundary conditons.

The above equation can be associated with a heat conduction equation [1, p. 130]

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - h(u - u_0), \tag{1·3}$$

provided $\qquad u_0 = 0 \qquad$ and $h = k\,x^2$.

2. We assume that our solution tends to zero for large values of t and when $|x| \to \infty$.

Let us substitute

$$u = A\,e^{-\lambda t}\,v(x) \text{ in } (1·2)$$

where λ is some positive constant, so that we obtain

$$\frac{d^2 v}{dx^2} + \left(\frac{\lambda}{k} - x^2\right)v = 0. \tag{2·1}$$

The equation (2·1) possesses solutions which tend to zero as $|x| \to \infty$ if and only if (λ/k) is of the form $1 + 2n$, where n is a positive integer [4, p. 155].

The solution of (2·1) is [4, p. 155]

(2·2) $\quad v = a_n \Psi_n(x) = a_n e^{-\frac{1}{2}x^2} H_n(x)$

where a_n is an arbitrary constant.

We may therefore assume the solution of (1·2) to be

$$u(x, t) = \sum_{n=0}^{\infty} a_n \; e^{-(1+2n)kt} \; \Psi_n(x). \tag{2·3}$$

When $t = 0$, let $u(x, 0) = f(x)$

then

$$f(x) = \sum_{n=0}^{\infty} a_n \; \Psi_n(x) \tag{2·4}$$

Now making use of the orthogonal property of $\Psi_n(x)$ we formally obtain,

$$a_n = \frac{1}{2^n \, n! \sqrt{\pi}} \int_{-\infty}^{\infty} f(x) \; \Psi_n(x) \, dx \tag{2·5}$$

Let $\qquad f(x) = e^{-\frac{1}{2}x^2} \; x^2$

Then because [3, p. 194]

$$x^2 = \sum_{r=0}^{1} \frac{2! \, H_{2-2r}(x)}{2^2 \, r! \, (2-2r)!}$$

in (2·3) and (2·4) we shall have $a_0 = \frac{1}{2}$, $a_1 = 0$, $a_2 = \frac{1}{4}$ and $a_n = 0$ for $n > 2$.

Let

$$\begin{aligned} f(x) &= e^{-\frac{1}{2}x^2} \, L_2(x) \\ &= e^{-\frac{1}{2}x^2} (1 - 2x + x^2/2) \end{aligned}$$

where $L_n(x)$ is the Laguerre polynomial, then because [3, p. 215]

$$L_2(x) = \sum_{k=0}^{2} {}_2F_2 \left[\begin{matrix} -\frac{1}{2}(2-k), -\frac{1}{2}(1-k) \, ; \\ \frac{1}{2}(1+k), \frac{1}{2}(2+k) \, ; \end{matrix} \; \frac{1}{4} \right] \frac{(-n)_k \, H_k(x)}{2^k \, (k!)^2},$$

we shall have

$$a_0 = {}_2F_2 \left[\begin{matrix} -1, -\frac{1}{2} \, ; \\ \frac{1}{2}, \; 1 \, ; \end{matrix} \; \frac{1}{4} \right] = \frac{17}{16}$$

$$a_1 = {}_2F_2 \left[\begin{matrix} -\frac{1}{2}, \; 0 \, ; \\ 1, \; \frac{3}{2} \, ; \end{matrix} \; \frac{1}{4} \right] = 1$$

$$a_2 = {}_2F_2 \left[\begin{matrix} 0, \; \frac{1}{2} \, ; \\ \frac{3}{2}, \; 2 \, ; \end{matrix} \; \frac{1}{4} \right] = 1.$$

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EFFECT OF MAGNETO-THERMO-ELASTIC INTERACTIONS ON THE COOLING PROCESS IN A SEMI-INFINITE SOLID

By

S. N. MURTHY

Department of Applied Mathematics, Shri Govindram Seksaria Technological Institute, Indore

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ABSTRACT

The problem of magneto-thermo-elastic interactions in a semi-infinite thermo-elastic solid embedded in a transverse magnetic field and subjected to a constant temperature throughout, is considered, when the bounding surface of the solid is suddenly cooled to zero temperature. To the first order terms of the thermoelastic coupling factor, it is found that the magnetic field has got the effect of sustaining the cooling process near the boundary and accelerating the same away from the boundary. The thermo-elastic coupling factor itself has got the effect of accelerating the cooling process throughout the solid.

INTRODUCTION

In recent years there has been much interest in the solution of problems in Magneto-thermo-elasticity. Paria (1962, 1964) has solved the problems of (*i*) Propagation of one dimensional magneto-thermo-elastic plane waves and (*ii*) Magneto-thermo-elastic interactions in an infinite solid due to instantaneous heat sources. The first problem has been generalised by Willson (1963). Kaliski and Nowacki (1962) have tackled the magneto-thermo-elastic problem of half-space under the action of a thermal shock on the bounding surface.

In the present paper we consider the problem of magneto-thermo-elastic interactions in a semi-infinite thermo-elastic solid embedded in a transverse magnetic field and subjected to constant temperature throughout, when the bounding surface of the solid is suddenly cooled to zero temperature. It is found that, for small values of the thermo-elastic-coupling factor, the magnetic field has got the effect of sustaining the cooling process in the solid near the boundary and accelerating the same away from the boundary ; while the thermo-elastic-coupling factor itself has got the effect of accelerating the cooling process throughout. It is assumed that the solid is a perfect conductor and that the radiation effects are negligible.

STATEMENT OF THE PROBLEM

A semi-infinite elastic solid bounded by the plane $x=0$ and extending in the positive direction of x–axis is kept initially at a constant temperature $T=T_0$ and it is also embedded in a constant magnetic field H_3 acting in positive direction of z–axis. The face $x=0$ which is rigidly fixed and kept at the initial magnetic field H_3 for all times is suddenly cooled to temperature $T=0$. We study the effects of the thermo-elastic-coupling factor ε_T and the magnetic field H_3 on the subsequent temperature distribution of the solid.

FUNDAMENTAL EQUATIONS

The following equations (Paria, 1964) are fundamental in the present context. Maxwell's equations in the absence of displacement current are given by :

$$\left.\begin{aligned} \text{curl}\ \vec{H} &= \vec{j}\ ,\ \text{div}\ \vec{B} = 0 \\ \text{curl}\ \vec{E} &= -\frac{\partial \vec{B}}{\partial t},\ \vec{B} = \mu_e\ \vec{H}, \end{aligned}\right\} \tag{3·1}$$

While Ohm's law may be written as

$$\vec{j} = \sigma\left(\vec{E} + \frac{\partial \vec{u}}{\partial t} \times \vec{B}\right) - K_0 \text{ grad } T, \tag{3·2}$$

where K_0 is the coefficient connecting the electric field with the temperature gradient. Hooke-Duhamel law, connecting the stress, strain and temperature is given by

$$\tau_{ij} = 2\mu\, e_{ij} + (\lambda e - \beta T)\, \delta_{ij}\,, \tag{3·3}$$

where

$$e_{ij} = \tfrac{1}{2}\,(u_{i,j} + u_{j,i}), \text{ and } e = \text{div } \vec{u}. \tag{3·4}$$

Energy equation is given by the Fourier's heat conduction equation in the generalised form as

$$K\,\nabla^2 T + Q = \rho\, c_v \frac{\partial T}{\partial t} + T_0\, \beta \frac{\partial e}{\partial t} + \pi_0 \text{ div } \vec{j}, \tag{3·5}$$

where π_0 is the coefficient connecting the current density vector with the vector of the density of heat flow. The equation of motion of the solid is given by

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \tau_{ik}}{\partial x_k} + \left(\vec{j} \times \vec{B}\right)_i \tag{3·6}$$

Now, substitution from (3·3) and (3·4) for τ_{ij} and e_{ij} in equation (3·6) gives

$$\mu\, \nabla^2 \vec{u} + (\lambda + \mu) \text{ grad } e - \beta \text{ grad } T + (\vec{j} \times \vec{B}) = \rho\, \frac{\partial^2 \vec{u}}{\partial t^2}. \tag{3·7}$$

CASE OF A PERFECT CONDUCTOR AND ONE DIMENSIONAL DEFORMATION

In what follows, we consider the case where the solid is a perfect conductor and the vector $\vec{u}$ has components (u, o, o) and all the other variables are functions of x and t only. We get from (3·1), the equations

$$j_x = 0,\, j_y = -\frac{\partial H_z}{\partial x}\ ,\, j_z = \frac{\partial H_y}{\partial x}. \tag{4·1}$$

$$\frac{\partial H_x}{\partial t} = \frac{\partial H_x}{\partial x} = 0, \frac{\partial E_z}{\partial x} = \mu_e \frac{\partial H_y}{\partial t}\ ,\ \frac{\partial E_y}{\partial x} = -\mu_e \frac{\partial H_z}{\partial t}\ . \tag{4·2}$$

It follows from (3·2), that

$$\left.\begin{aligned} E_x &= 0, \\ E_y &= \mu_e H_z \frac{\partial u}{\partial t}, \\ E_z &= -\mu_e H_y \frac{\partial u}{\partial t}, \end{aligned}\right\} \tag{4·3}$$

if we assume the conductivity to be infinite *i.e.*, $\sigma \to \infty$.

The last two equations of (4·3) can respectively be written by using the last two equations of (4·2) as

$$\frac{\partial H_y}{\partial t} = -\frac{\partial}{\partial x}\left(H_y \frac{\partial u}{\partial t}\right), \tag{4·4}$$

$$\frac{\partial H_z}{\partial t} = -\frac{\partial}{\partial x}\left(H_z \frac{\partial u}{\partial t}\right). \tag{4·5}$$

Eleminating $\vec{j}$ from (3·7) with the help of (4·1), we get,

$$(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial T}{\partial x} - \tfrac{1}{2}\mu_e \frac{\partial}{\partial x}\left(H_y^2 + H_z^2 \right) = \rho \frac{\partial^2 u}{\partial t^2}. \tag{4·6}$$

Now we linearize the equations (4·4), (4·5), and (4·6) by putting $H_y = h_y$, $H_z = H_3 + h_z$, where h_y and h_z are so small that their squares and higher powers as well as their products with u can be neglected. Thus we obtain :

$$\frac{\partial h_y}{\partial t} = 0, \tag{4·7}$$

$$\frac{\partial h_z}{\partial t} = -H_3 \frac{\partial^2 u}{\partial x \partial t}, \tag{4·8}$$

$$(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial T}{\partial x} - \mu_e H_3 \frac{\partial h_z}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}. \tag{4·9}$$

Heat conduction equation (3·5) can be written for the case under consideration (one dimensional form), when there are no heat sources, as

$$K \frac{\partial^2 T}{\partial x^2} = \rho c_v \frac{\partial T}{\partial t} + T_0 \beta \frac{\partial^2 u}{\partial x\, \partial t}. \tag{4·10}$$

Equation (4·7) gives $h_y = 0$, since h_y is zero at time $t=0$.

SOLUTION OF THE PROBLEM

Applying Laplace transform to equations (4·8) – (4·10) and denoting the Laplace transform of a variable by a bar over it and assuming that all the variables are initially zero, except T which is T_0 when $t=0$, we get respectively the equations,

$$\frac{d\bar{u}}{dx} = -\frac{\bar{h}_z}{H_3}, \tag{5·1}$$

$$(\lambda + 2\mu) \frac{d^2\bar{u}}{dx^2} - \rho \; p^2 \bar{u} = \beta \frac{d\bar{T}}{dx} + \mu_e H_3 \frac{d\bar{h}_z}{dx}, \tag{5·2}$$

$$K \frac{d^2\bar{T}}{dx^2} - \rho \; c_v p \; \bar{T} = T_0 \; \beta p \frac{d\bar{u}}{dx} - \rho \; c_v \; T_0. \tag{5·4}$$

Substitution for $\frac{d\bar{u}}{dx}$ from (5·1) in (5·3) gives :

$$K \frac{d^2\bar{T}}{dx^2} - \rho \; c_v \; p \; \bar{T} = - \frac{T_0 \beta \; p \bar{h}_z}{H_3} - \rho \; c_v \; T_0. \tag{5·4}$$

We obtain, by differentiating (5·2) with respect to x, and substituting for $\frac{d^3\bar{u}}{dx^3}$ and $\frac{d\bar{u}}{dx}$ with the help of (5·1) the equation,

$$-\frac{d^2\bar{h}_z}{dx^2}(1 + R_H) + \frac{p^2}{c_1^2} \bar{h}_z = \frac{H_3 \beta}{\rho \; c_1^2} \frac{d^2\bar{T}}{dx^2}, \tag{5·5}$$

where $\frac{\lambda + 2\mu}{\rho} = c_1^2$ and $\frac{\mu_e \; H_3^2}{\rho \; c_1^2} = R_H$. Multiplying (5·4) by $\frac{H_3 \; \beta}{\rho \; c_1^2}$ and (5·5) by K and subtracting the former from the latter, we have

$$\bar{T} = - \frac{c_1^2 \; K}{H_3 \; \beta c_v p} (1 + R_H) \frac{d^2\bar{h}_z}{dx^2} + \frac{c_1^2}{H_3 \; \beta c_v p} \left(\frac{K \; p^2}{c_1^2} + \frac{T_0 \; \beta^2 \; p}{\rho \; c_1^2} \right) \bar{h}_z + \frac{T_0}{p}. \tag{5·6}$$

Substitution for $\bar{T}$ from (5·6) in (5·4) gives

$$K_1 \frac{d^4\bar{h}_z}{dx^4} - \left(\frac{K_1 p^2}{c_3^2} + \frac{\varepsilon_T \; p}{1 + R_H} + p \right) \frac{d^2\bar{h}_z}{dx^2} + \frac{p^3}{c_3^2} \bar{h}_z = 0, \tag{5·7}$$

where $c_1^2(1 + R_H) = c_3^2$, $\varepsilon_T = \frac{T_0 \; \beta^2}{\rho^2 \; c_v \; c_1^2}$, and $K_1 = \frac{K}{\rho c_v}$.

Solving (5·7), we get

$$\bar{h}_z = C_1 \; e^{-m_1 x} + C_2 \; e^{-m_2 x}, \tag{5·8}$$

where m_1, m_2 are the roots with positive real parts of the equation

$$K_1 \; n^2 - \left(p + \frac{K_1 p^2}{c_3^2} + \frac{p \varepsilon_T}{1 + R_H} \right) n + \frac{p^3}{c_3^2} = 0, \; (n = m^2) \tag{5·9}$$

and C_1, C_2 are arbitrary constants. From equation (5·1), we get, by using (5·8)

$$\bar{u} = \frac{1}{H_3} \left(\frac{C_1 \; e^{-m_1 x}}{m_1} + \frac{C_2 \; e^{-m_2 x}}{m_2} \right) + C_3, \tag{5·10}$$

where C_3 is another arbitrary constant. Also we get from (5·6) using (5·8)

$$\bar{T} = - \frac{K \; c_3^2}{H_3 \; \beta c_v \; p} \left(C_1 \; m_1^2 \; e^{-m_1 x} + C_2 \; m_2^2 \; e^{-m_2 x} \right) + \\ + \frac{1}{H_3 \; \beta c_v} \left(Kp + \frac{T_0 \; \beta^2}{\rho} \right) \left(C_1 \; e^{-m_1 x} + C_2 \; e^{-m_2 x} \right) + \frac{T_0}{p}. \tag{5·11}$$

The boundary conditions of the problem can be expressed as $u\ (o,\ t)=0$, $H_z\ (o,\ t)=H_3$, *i.e.* $h_z\ (o,\ t)=0$, and $T\ (o,\ t)=0$. In terms of the Laplace transforms, the above conditions become $\bar{u}\ (p)=0$, $\bar{h}_z\ (p)=0$ and $\bar{T}\ (p)=0$. Using the above boundary conditions we have for C_1, C_2, C_3, the values

$$\left.\begin{aligned} C_1 &= -\frac{H_3\,\beta c_v\,T_0}{K\,c_3^2\,(m_2^2-m_1^2)}, \quad C_2 = -\frac{H_3\,\beta c_v\,T_0}{K\,c_3^2\,(m_1^2-m_2^2)}, \\ \text{and} \quad C_3 &= -\frac{\beta c_v\,T_0}{K\,c_3^2\,m_1 m_2\,(m_1+m_2)}. \end{aligned}\right\} \tag{5·12}$$

Substituting the values of C_1 and C_2 in (5·11), we get

$$\bar{T} = \frac{T_0}{p(m_2^2-m_1^2)}\left\{(m_1^2-\xi)\,e^{-m_1x}-(m_2^2-\xi)\,e^{-m_2x}\right\}+\frac{T_0}{p}, \tag{5·13}$$

where

$$\xi = \left(K\,p^2+\frac{T_0\,\beta^2\,p}{\rho}\right)\frac{1}{K\,c_3^2}. \tag{5·14}$$

From (5·9), we have

$$n = \frac{p}{2K_1}\left\{\left(1+\frac{\varepsilon_T}{1+R_H}+\frac{K_1\,p}{c_3^2}\right)\pm\sqrt{\left(\frac{K_1\,p}{c_3^2}+\frac{\varepsilon_T}{1+R_H}-1\right)^2+\frac{4\varepsilon_T}{1+R_H}}\right\}.$$

When $\varepsilon_T\to 0$, and $R_H\to 0$,

$$n = \frac{p}{2K_1}\left\{\left(1+\frac{K_1\,p}{c_3^2}\right)\pm\left(\frac{K_1\,p}{c_3^2}-1\right)\right\},$$

so that we get $m_1^2 = \dfrac{p^2}{c_3^2}$, $m_2^2 = \dfrac{p}{K_1}$. Also for non-elastic rigid bodies, $c_1^2 = \dfrac{\lambda+2\mu}{\rho}\to\infty$ and so $m_1^2\to 0$ and equation (5·7) simplifies to $\bar{T} = -\dfrac{T}{p}\,e^{-\sqrt{p/K_1}x}+\dfrac{T_0}{p}$, which is the classical result [Carslaw and Jaeger (1959)] in the form of Laplace transform.

Substituting for C_1, C_2 and C_3 from (5·12) in equations (5·8) and (5·10), we get

$$\bar{h}_z = \frac{H_3\,\beta c_v\,T_0}{K\,c_3^2\,(m_1^2-m_2^2)}\left(e^{-m_1x}-e^{-m_2x}\right) \tag{5·15}$$

and

$$\bar{u} = \frac{\beta\,c_v\,T_0}{K\,c_3^2\,m_1\,m_2\,(m_1^2-m_2^2)}\left(m_2\,e^{-m_1x}-m_1\,e^{-m_2x}+m_1-m_2\right) \tag{5·16}$$

APPROXIMATION FOR SMALL VALUES OF THERMO-ELASTIC-COUPLING FACTOR

Now from equation (5·9), we have

$$\left.\begin{aligned} m_1^2+m_2^2 &= \frac{p}{K_1\,c_3^2}\,(K_1\,p+c_1^2\,\varepsilon_T+c_3^2), \\ \text{and} \quad m_1\,m_2 &= \frac{1}{c_3}\sqrt{\frac{p_3}{K_1}}. \end{aligned}\right\} \tag{6·1}$$

From (6·1). we can write

$$m_1 + m_2 = \left\{\frac{p}{K_1\ c_3^2}\,(K_1 p + c_1^2\ \varepsilon_T\ + c_3^2) + \frac{2}{c_3}\sqrt{\frac{p^3}{K_1}}\right\}^{\frac{1}{2}},$$

and

$$m_1 - m_2 = \left\{\frac{p}{K_1\ c_3^2}\ \ (K_1 p + c_1^2\ \varepsilon_T\ + c_3^2) - \frac{2}{c_3}\sqrt{\frac{p^3}{K_1}}\right\}^{\frac{1}{2}}.$$

Assuming that the thermo-elastic-coupling factor ε_T is small and expanding the above equations in powers of ε_T and neglecting squares and higher powers of ε_T, we obtain

$$m_1 + m_2 = \frac{1}{c_3}\sqrt{\frac{p}{K_1}}\,(\sqrt{K_1 p} + c_3)\left\{1 + \tfrac{1}{2}\ \frac{c_1^2\ \varepsilon_T}{(\sqrt{K_1 p} + c_3)^2}\right\}$$

$$m_1 - m_2 = \frac{1}{c_3}\sqrt{\frac{p}{K_1}}\,(\sqrt{K_1 p} - c_3)\left\{1 + \tfrac{1}{2}\ \frac{c_1^2\ \varepsilon_T}{(\sqrt{K_1 p} - c_3)^2}\right\},$$

From the above equations, we get

$$m_1^2 - m_2^2 = \frac{p}{K_1\, c_3^2}\,(K_1 p - c_3^2)\ \left\{\ 1 + \frac{c_1^2\,\varepsilon_T\,(K_1 p + c_3^2)}{(K_1 p - c_3^2)^2}\right\}, \tag{6·2}$$

$$m_1 = \frac{p}{c_3}\left(1 + \tfrac{1}{2}\,\frac{c_1^2\ \varepsilon_T}{K_1 p - c_3^2}\right), \text{ and } \quad m_2 = \sqrt{\frac{p}{K_1}}\left(1 - \tfrac{1}{2}\,\frac{c_1^2\ \varepsilon_T}{K_1 p - c_3^2}\right). \tag{6·3}$$

Also, we have from equations (6·1) and (6·2),

$$m_1^2 = \frac{p^2}{c_3^2}\left(\ 1 + \frac{c_1^2\ \varepsilon_T}{K_1 p - c_3^2}\ \right),\ m_2^2 = \frac{p}{K_1}\left(\ 1 - \frac{c_1^2\ \varepsilon_T}{K_1 p - c_3^2}\ \right). \tag{6·4}$$

We obtain from equations (5·14), (6·2) and (6·4)

$$\frac{\xi - m_1^2}{m_1^2 - m_2^2} = -\frac{c_1^2\ c_3^2\ \varepsilon_T}{(K_1 p - c_3^2)^2}\ ,\ \frac{\xi - m_2^2}{m_1^2 - m_2^2}\ =\ 1 - \frac{c_1{}^2\ c_3{}^2\ \varepsilon_T}{(K_1 p - c_3{}^2)^2}$$

Using the above equations in (5·13), $\bar{T}$ can be written as

$$\bar{T} = \frac{T_0}{p} + \frac{T_0}{p}\ \left\{\frac{c_1{}^2\ c_3{}^2\ \varepsilon_T}{(K_1 p - c_3{}^2)^2}\ e^{-\frac{p}{c_3}x} - \left(1 - \frac{c_1{}^2\ c_3{}^2\ \varepsilon_T}{(K_1 p - c_3{}^2)^2}\right)\ e^{-\sqrt{p/k_1}x} - \right.$$

$$\left. - \tfrac{1}{2}\ \frac{c_1{}^2\ \varepsilon_T}{K_1 p - c_3{}^2}\ \sqrt{\frac{p}{K_1}}\,x\,e^{-\sqrt{p/k_1}\,x}\ \right\}. \tag{6·5}$$

Taking the inverse Laplace transform of the above equation, by using tables [Erdelyi (1954),] and introducing the dimensionless quantities,

$$\eta = \frac{x c_1}{K_1}\ ,\ \tau = c_1\sqrt{\frac{t}{K_1}}, \text{ we obtain}$$

$$\frac{T}{T_0} = \left\{1 - \operatorname{erfc}\left(\frac{\eta}{2\tau}\right)\right\}\ -\ \varepsilon_T\ \ F(\eta,\ \tau,\ R_M). \tag{6·6}$$

In equation (6·6),

$$R_M = 1 + R_H$$

and

$$F(\eta, \tau, R_M) = I(\tau^2\sqrt{R_M} - \eta) \left\{ \frac{1}{R_M} - \left(\frac{1}{R_M} + \frac{\eta}{\sqrt{R_M}} - \tau^2 \right) \exp\left(\tau^2 R_M - \eta\sqrt{R_M}\right) \right\} -$$

$$- \frac{1}{R_M} \operatorname{erfc}\left(\frac{\eta}{2\tau}\right) + \tfrac{1}{2}\left(\frac{1}{R_M} + \frac{\eta}{\sqrt{R_M}} - \tau^2\right) \exp\left(\tau^2 R_M - \eta\sqrt{R_M}\right)$$

$$\operatorname{erfc}\left(\frac{\eta}{2\tau} - \tau\sqrt{R_M}\right) + \tfrac{1}{2}\left(\frac{1}{R_M} - \frac{\eta}{\sqrt{R_M}} - \tau^2\right) \exp\left(\tau^2 R_M + \eta\sqrt{R_M}\right)$$

$$\operatorname{erfc}\left(\frac{\eta}{2\tau} + \tau\sqrt{R_M}\right),$$

where

$$I(\tau^2\sqrt{R_M} - \eta) = 0, \; \tau^2 < \frac{\eta}{\sqrt{R_M}}$$

$$= 1, \; \tau^2 > \frac{\eta}{\sqrt{R_M}}.$$

In equation (6·6) which gives the temperature of the solid the first term within brackets, represents the classical result [Carslaw and Jaeger (1959)] while the second term is the contribution due to the thermo-elastic-coupling factor ε_T and it also contains the effect of the initial magnetic field H_3 represented by R_M. It can be seen that when $\eta = 0$ the perturbation function $F(\eta, \tau, R_M)$ in (6·6) vanishes. Also when $\eta \to \infty$ it can be seen that $F(\eta, \tau, R_M) \to 0$, for in that case

$$I(\tau^2\sqrt{R_M} - \eta) = 0, \; \operatorname{erfc}\left(\frac{\eta}{2\tau}\right) \to 0$$

and then

$$\lim_{\eta \to \infty} F(\eta, \tau, R_M) = \lim_{\eta \to \infty} \left[\tfrac{1}{2}\left(\frac{1}{R_M} + \frac{\eta}{\sqrt{R_M}} - \tau^2\right) \exp\left(\tau^2 R_M - \eta\sqrt{R_M}\right) \right.$$

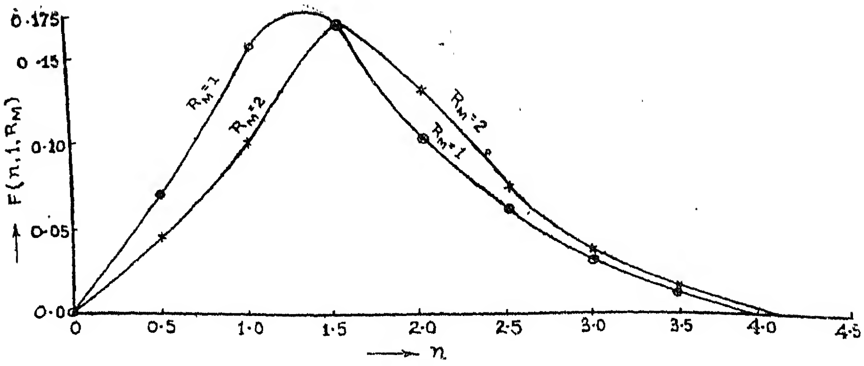
$$\left. \operatorname{erfc}\left(\frac{\eta}{2\tau} - \tau\sqrt{R_M}\right) + \tfrac{1}{2}\left(\frac{1}{R_M} - \frac{\eta}{\sqrt{R_M}} \tau^2\right) \exp\left(\tau^2 R_M + \eta\sqrt{R_M}\right) \operatorname{erfc}\left(\frac{\eta}{2\tau} + \tau\sqrt{R_M}\right) \right]$$

$$= \lim_{\eta \to \infty} \left[\frac{\eta}{2\sqrt{R_M}} \operatorname{erfc}\left(\frac{\eta}{2\tau}\right) \left\{ \exp\left(-\eta\sqrt{R_M}\right) - \exp\left(\eta\sqrt{R_M}\right) \right\} \right].$$

By using the asymptotic expansion $\operatorname{erfc}\left(\frac{\eta}{2\tau}\right) = \frac{e^{-\eta^2/4\tau^2}}{\sqrt{\pi}\,\eta/2\tau}$, for large η, it can be shown that the above limit tends to zero. Thus we have

$$\lim_{\eta \to \infty} F(\eta, \tau, R_M) = 0.$$

The following table gives the values of $F(\eta, \tau, R_M)$ for $\tau = 1$ and $R_M = 1, 2$ for various values of η. The graphs in the figure give the variation of $F(\eta, 1, 1)$ and $F(\eta, 1, 2)$ with η.



TABLE

η	$F(\eta, 1, 1)$	$F(\eta, 1, 2)$
0·0	0·0000	0·0000
0·5	0·0708	0·0448
1·0	0·1555	0·0987
1·5	0·1699	0·1710
2·0	0·1162	0·1323
2·5	0·0619	0·0742
3·0	0·0298	0·0375
3·5	0·0121	0·0163

From the figure it can be seen that the perturbation function $F(\eta, 1, 1)$ in the absence of the magnetic field monotonically increases as η increases from 0·0 to 1·3 and then monotonically decreases for all values of $\eta > 1·3$. The perturbation function $F(\eta, 1, 2,)$ in the presence of the magnetic field, monotonically increases as η increases from 0·0 to 1·5 and then monotonically decreases for all values of $\eta > 1·5$. From this and the equation (6·6) it can be concluded that the cooling effect of the thermo-elastic-coupling factor gradually increases in the portion of the solid near the boundary and then it decreases gradually in the remaining portion.

Also it can be observed from the figure that $F(\eta, 1, 2)$ is less than $F(\eta, 1, 1)$ for values of η between 0 and 1·5 and that $F(\eta, 1, 2)$ is greater than $F(\eta, 1, 1)$ for values of $\eta > 1·5$. This means that the effect of the magnetic field is to sustain the cooling process in the portion of the solid which is near the boundary and to accelerate the same in the rest of the solid.

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THE SPECTRUM OF BaBr MOLECULE IN THERMAL EMISSION

By

R. K. MISHRA *and* K. MAJUMDAR

Physics Department, Allahabad University

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ABSTRACT

Investigation of the spectrum of BaBr molecule has been made in thermal emission for the first time. A large number of new violet degraded bands along with the known ones have been recorded in the near ultraviolet region. All these bands have been classified into systems $D \rightarrow X$ ($\lambda\lambda$ 4000–3740) and $E \rightarrow X$ ($\lambda\lambda$ 3810–3580) represented by the following expressions suggested by Harrington (1942).

$$D \rightarrow X, \quad \nu = 25670{\cdot}93 + 209{\cdot}11\,(v' + \tfrac{1}{2}) - 0{\cdot}53\,(v' + \tfrac{1}{2})^2 - 194{\cdot}14\,(v'' + \tfrac{1}{2}) + 0{\cdot}42\,(v'' + \tfrac{1}{2})^2.$$

$$E \rightarrow X, \quad \nu = 26865{\cdot}7 + 2199{\cdot}1\,(v' + \tfrac{1}{2}) - 0{\cdot}35\,(v' + \tfrac{1}{2})^2 - 193{\cdot}91\,(v'' + \tfrac{1}{2}) + 0{\cdot}45\,(v'' + \tfrac{1}{2})^2.$$

INTRODUCTION

The band spectrum of the diatomic molecule BaBr has been studied by a number of investigators. Olmsted (1906) was the first to observe a few bands in visible and ultraviolet regions in flame emission. Walters and Barratt (1929) investigated the absorption spectrum and photographed only seven bands in the green region. Hedfeld (1931) obtained a larger number of bands in the region $\lambda\lambda$ 5410 – 5100 using an arc and acetylene flame fed with barium bromide and classified them into one system. A closer investigation of the absorption spestrum of the molecule was undertaken by Harrington (1942), who observed bands in the ultraviolet region also and classified them into two systems $D \leftarrow X$ and $E \leftarrow X$ in the regions $\lambda\lambda$ 3940–3740 and $\lambda\lambda$ 3970–3650. He also reanalysed the visible system, *i.e.* $C \rightleftharpoons X$, obtained by Hedfeld (1931), It is found that the bands of the $C \rightleftharpoons X$ system are degraded towards longer wavelength, that of $D \leftarrow X$ and $F \leftarrow X$ system are degarded towards shorter wavelength. Although all the three systems have been studied in absorption, only the system $C \rightleftharpoons X$ has been obtained in emission as well by the previous workers. It was with a view to observe the system $D \rightarrow X$ and $E \rightarrow X$ in emission that the present investigations were carried out. The authors have studied in detail the spectrum of BaBr molecule in thermal emission and have been able to record a large number of new bands along with the known ones in the regions $\lambda\lambda$ 4000 – 3740 and $\lambda\lambda$ 3810 – 3580. All these observed bands have been fitted into the vibrational schemes of $D \rightarrow X$ and $E \rightarrow X$ systems.

EXPERIMENTAL

The investigations were carried out with the vacuum graphite tube furnace of this laboratory described in detail by H. Mohan and K. Majumdar (1961). A few grams of Barium Bromide (BDH) were put into the tube of the furnace which was 9 cm long and had an internal diameter of 0·8 cm. The rear end of the experimental tube was closed with a graphite stopper and the furnace chamber was evacuated. Nitrogen gas at a pressure of 45 cm of mercury was filled inside the furnace chamber with a view to reduce rapid vaporisation of the substance from the experimental tube. A temperature of the order of 2200°C - 2400°C was found suitable for obtaining good spectrograms. The spectrum was photographed on Hilger E492 large quartz spectrograph which had a reciprocal dispersion of about 12 A/mm near λ 4000. Ilford R 40 Rapid Process Panchromatic plates were used and exposures of nearly two minutes were sufficient for recording a good photograph. Copper arc spectrum was used as the comparison standard.

RESULTS

The wavelengths of the bandheads and visual estimates of their intensities have been listed in tables I and II. All the observed bands have been fitted into the systems $D \to X$ and $E \to X$ and can be represented by the following expressions proposed by Harrington (1942).

$$D \to X \text{ system}, \quad \nu = 25670{\cdot}93 + 209{\cdot}14\,(v' + \tfrac{1}{2}) - 0{\cdot}53\,(v' + \tfrac{1}{2})^2 - 194{\cdot}14\,(v'' + \tfrac{1}{2}) + 0{\cdot}42\,(v'' + \tfrac{1}{2})^2.$$

$$E \to X \text{ system}, \quad \nu = 26865{\cdot}7 + 219{\cdot}91\,(v' + \tfrac{1}{2}) - 0{\cdot}35\,(v' + \tfrac{1}{2})^2 - 193{\cdot}91\,(v'' + \tfrac{1}{2}) + 0{\cdot}45\,(v'' + \tfrac{1}{2})^2.$$

Calculated and observed wavenumbers and vibrational classifications have also been entered in the above tables. Bands observed by Harrington (1942) have been marked with an asterisk (*). The reproduction of the spectrum has been given in the plate I. Deslandre's arrangements of the bands and the variations of the intensities for the two systems have been depicted in tables III and IV and figures II and III respectively. It will be noticed that all the intense bands lie on smooth Condon parabolas.

TABLE I

λ air	Int.	ν_{vac} obs.	ν_{vac} calc.	Classification
3982·8	2	25101	25101	(0, 3)
3979·9	4	25119	25119	(1, 4)
3977·2	3	25136	25136	(2, 5)
3974·5	2	25153	25153	(3, 6)
3971·8	1	25170	25169	(4, 7)
3952·5	5	25293	25293	(0, 2)
3950·0	4	25309	25309	(1, 3)
3947·4	3	25326	25326	(2, 4)
3944·9	2	25342	25342	(3, 5)
3942·5	1	25357	25358	(4, 6)
3940·1	1	25373	25373	(5, 7)
3922·6	6	25486	25485	(0, 1)*
3920·1	6	25502	25501	(1, 2)
3917·9	5	25517	25516	(2, 3)
3916·1	4	25528	25532	(3, 4)
3913·4	2	25546	25547	(4, 5)
3911·2	1	25560	25561	(5, 6)
3893·3	10	25679	25679	(0, 0)*
3890·8	0	25694	25693	(1, 1)*
3888·9	8	25707	25708	(2, 2)*

λ air	Int.	ν_{vac} obs.	calc.	Classification
3886·6	1	25723	25722	(3, 3)*
3883·9	4	25740	25736	(4, 4)
3882·2	1	25751	25750	(5, 5)*
3861·8	7	25887	25887	(1, 0)*
3859·9	6	25900	25900	(2, 1)*
3857·8	6	25914	25914	(3, 2)*
3855·8	4	25927	25927	(4, 3)*
3853·8	2	25941	25940	(5, 4)
3831·2	6	26094	26094	(2, 0)*
3829·4	6	26106	26106	(3, 1)*
3827·7	5	26118	26119	(4, 2)*
3825·9	2	26130	26130	(5, 3)*

TABLE II

3815·4	2	26202	26203	(3, 7)
3810·9	2	26233	26233	(4, 8)
3806·4	2	26264	26264	(5, 9)
3801·3	5	26302	26302	(0, 3)
3796·9	5	26331	26332	(1, 4)
3792·5	4	26360	26361	(2, 5)
3788·2	3	26390	26390	(3, 6)
3784·1	2	26419	26420	(4, 7)
3779·6	1	26450	26449	(5, 8)
3773·4	6	26494	26494	(0, 2)*
3769·3	5	26523	26522	(1, 3)*
3765·0	4	26553	26550	(2, 4)*
3761·3	4	26579	26579	(3, 5)*
3757·2	3	26608	26607	(4, 6)*
3753·2	1	26636	26636	(5, 7)*
3746·2	6	26686	26686	(0, 1)*
3742·4	6	26713	26713	(1, 0)
3738·2	1	26743	26741	(2, 3)
3734·4	0	26770	26768	(3, 4)
3730·5	0	26798	26796	(4, 5)
3727·1	0	26823	26824	(5, 6)
3722·9	0	26853	26852	(6, 7)*

*Bands observed by Harrington, (1942)

λ air	Int.	ν_{vac}		Classification
		obs.	calc.	
3719·3	8	26879	26879	(0, 0)*
3715·7	6	26905	26905	(1, 1)
3712·1	6	26931	26931	(2, 2)*
3708·4	4	26958	26958	(3, 3)
3704·7	3	26985	26985	(4. 4)
3701·1	2	27011	27012	(5, 5)
3697·4	1	27038	27039	(6, 6)
3695·0	4	27056	27066	(7, 7)
3689·2	6	27098	27098	(1, 0)*
3685·7	6	27124	27124	(2, 1)*
3682·2	5	27150	27150	(3, 2)
3678·7	4	27176	27176	(4, 3)
3675·2	4	27201	27202	(5, 4)
3671·8	3	27227	27228	(6, 5)
3668·1	2	27254	27254	(7, 6)
3664·5	1	27281	27281	(8, 7)
3659·7	5	27317	27317	(2, 0)
3656·5	5	27341	27342	(3, 1)*
3653·1	4	27366	27367	(4, 2)*
3649·6	4	27392	27392	(5, 3)
3646·5	3	27416	27416	(6, 4)
3642·9	2	27443	27443	(7, 5)
3639·9	1	27468	27469	(8, 6)
3630·7	2	27535	27535	(3, 0)
3627·3	4	27561	27559	(4, 1)
3624·4	4	27583	27583	(5, 2)
3621·0	3	27609	27608	(6, 3)
3618·2	3	27630	27631	(7, 4)
3615·5	3	27651	27656	(8, 5)
3611·4	2	27682	27682	(9, 6)
3608·0	1	27708	27707	(10, 7)
3602·6	1	27750	27752	(4, 0)
3599·2	1	27776	27776	(5, 1)
3596·3	3	27798	27799	(6, 2)
3593·2	3	27822	27823	(7, 3)
3590·0	2	27847	27846	(8, 4)
3586·4	1	27875	27871	(9, 5)

*Bands observed by Harrington, (1942)

TABLE III

v' \ v''	0	1	2	3	4	5	6	7
5				26130	25941	25751	25560	25373
4			26118	25927	25740	25546	25357	25170
3		26106	25914	25723	25528	25342	25153	
2	26094	25900	25707	25517	25326	25136		
1	25887	25694	25502	25309	25119			
0	25679	25486	25293	25101				

TABLE IV

v' \ v''	0	1	2	3	4	5	6	7	8	9
10								27708		
9						27875	27682			
8					27847	27651	27468	27281		
7				27822	27620	27443	27254	27056		
6			27798	27609	27416	27227	27038	26852	26665	
5		27776	27583	27392	27201	27011	26823	26636	26450	26264
4	27750	27561	27366	27176	26985	26798	26608	26419	26223	
3	27535	27341	27150	26958	26770	26578	26390	26202		
2	26317	27124	26931	26743	26553	26360	26172			
1	27098	26906	26713	26523	26331	26142				
0	26879	26686	26494	26302						

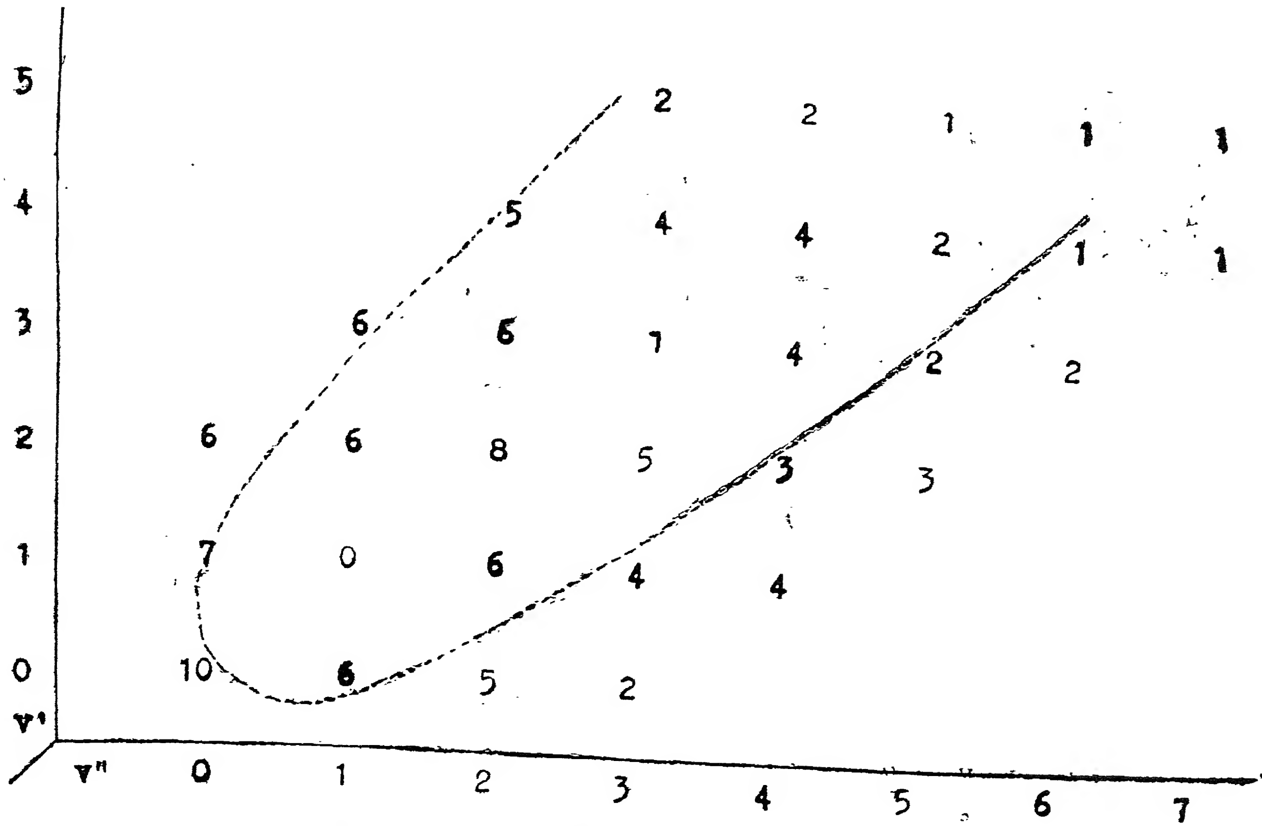


Fig. 11. Intensity Distribution in $D \rightarrow X$ system of BaBr

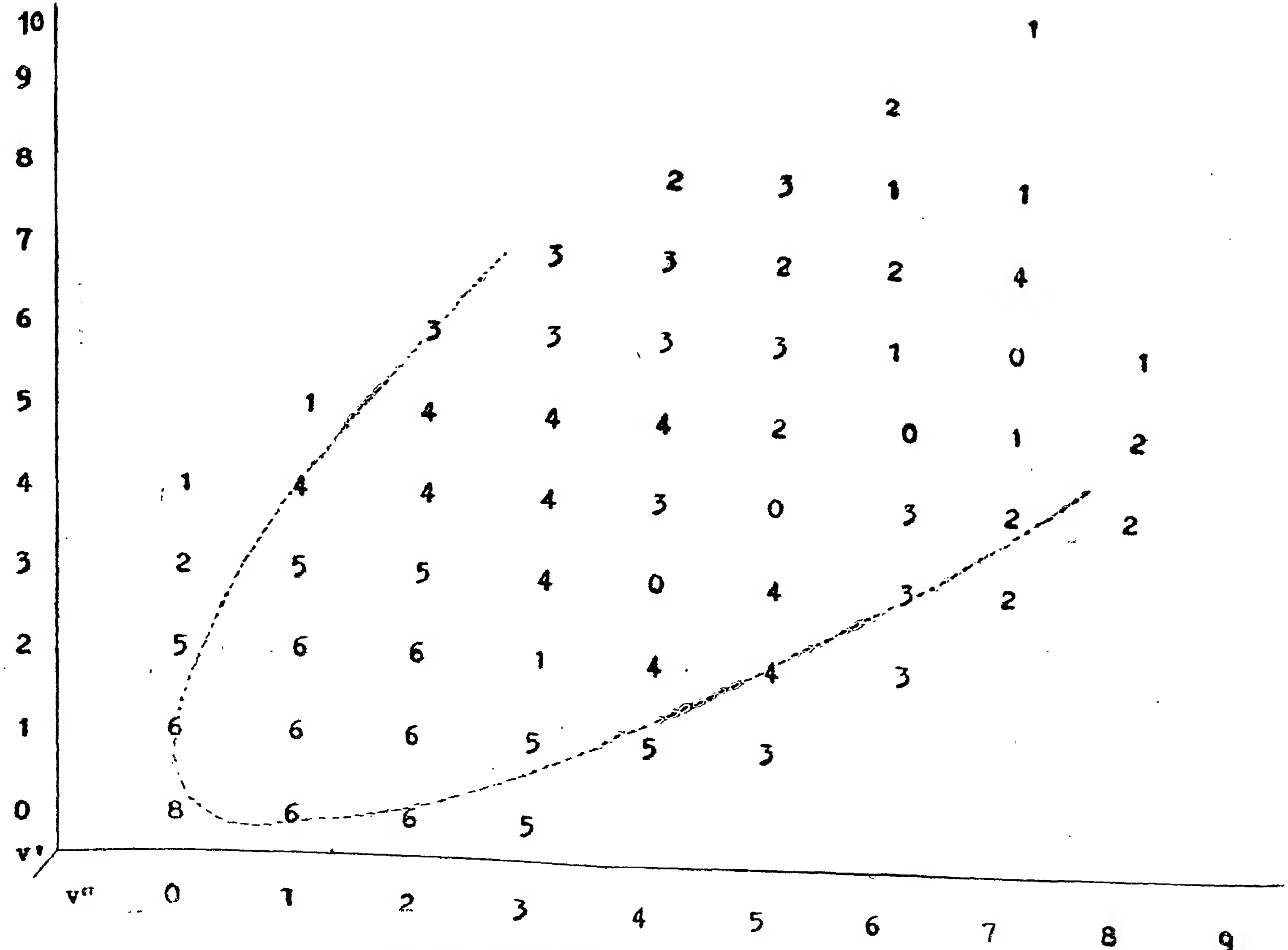
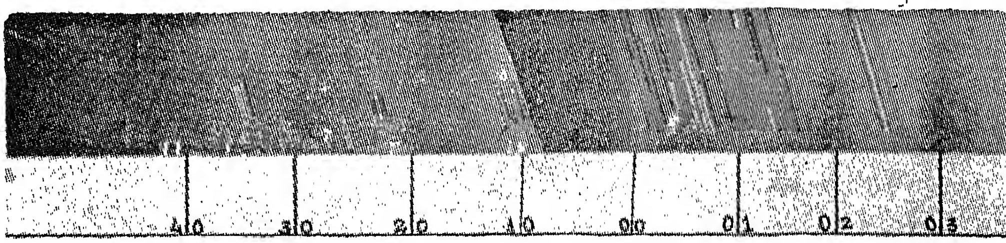


Fig. III. Intensity Distribution in $E \rightarrow X$ system of BaBr

PLATE 1

$E \rightarrow X$ System



$D \rightarrow X$ System

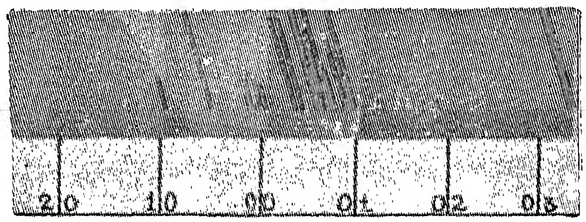


Fig. 1. Thermal Emission Spectrum BaBr.

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RELATIONS BETWEEN FUNCTIONS CONTIGUOUS TO CERTAIN HYPERGEOMETRIC FUNCTIONS OF THREE VARIABLES*

By

H. M. SRIVASTAVA

Department of Mathematics, University of Jodhpur, Jodhpur

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ABSTRACT

To the Lauricella's set of hypergeometric functions of three variables the author has recently added three new and distinct functions, namely H_A, H_B and H_C defined by means of the triple series

$$H_A(a, b, b'; c, c'; x, y, z)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(a)_{m+p} (b)_{m+n} (b')_{n+p}}{(c)_m (c')_{n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \tag{1}$$

$$H_B(a, b, b'; c_1, c_2, c_3; x, y, z)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(a)_{m+p} (b)_{m+n} (b')_{n+p}}{(c_1)_m (c_2)_n (c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \tag{2}$$

and

$$H_C(a, b, b'; c; x, y, z)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(a)_{m+p} (b)_{m+n} (b')_{n+p}}{(c)_{m+n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \tag{3}$$

respectively, where, for convergence,

$$|x| < R, \; |y| < S, \; |z| < T$$

such that

$$R + S + T = 1 + ST \text{ in (1)},$$

$$R + S + T + 2\sqrt{RST} = 1 \text{ for (2)}$$

and

$$R = S = T = 1 \text{ in case of (3)}.$$

These functions do not appear to be covered by Lauricella's conjecture nor does their existence seem to have been noticed earlier. It may be of interest to note that H_C is a generalisation of Appell's F_1, H_B generalises the Appell function F_2 while H_A furnishes a generalisation of both F_1 and F_2.

The present paper is devoted to a systematic study of the functions contiguous to these three hypergeometric functions of three variables and also to the derivation of the various recurrence relations.

*This paper was read by the author at the thirty-fifth annual session of the National Academy of Sciences of India.

1. INTRODUCTION

Two hypergeometric functions of three variables, *viz.* H_A and H_B, were recently defined by me ([6], p. 97 ; *see* also [8], p. 69 and [9], pp. 12–13). In a subsequent paper[7] I gave their integral representations of the various types and made use of the contour integrals of Pochhammer type in a detailed study of the transformation theory and the systems of partial differential equations associated with them. For instance, it was shown that [7, § 8]

$$(1\cdot1)\qquad H_A\,(a+a'-1, b, b'\,;\, a, b'\,;\, x, y, z)$$

$$= (1-x-y)^{-b}\,(1-z)^{-a'}\; G_2\left(b, a', 1-a, 1-a'\,;\, \frac{x}{1-x}, \frac{z}{1-z}\right)$$

$$(1\cdot2)\qquad H_A(a+a'-1, b, b'\,;\, a, b'\,;\, x, y, z)$$

$$= (1-x-y)^{-b}\, H_2\left(1-a, b, a+a'-1, a', a\,;\, \frac{x}{x-1}, -z\right)$$

where G_2 and H_2 are the double hypergeometric series defined by Horn[3], and that the differential system of

$$H_A(a, b, b'\,;\, c, c'\,;\, x, y, z)$$

possesses a general solution in the form [7, § 6]

$$(1\cdot3)\qquad W = \int_C t^{-\rho}\,(1-t)^{-\rho'}\, f\left(\frac{x}{t}, \frac{z}{1-t}, y\right)dt$$

$$(\rho+\rho' = 1+a)$$

whenever C is either a closed contour in the t-plane or an open contour at the two ends of which both

$$t^{-\rho}\,(1-t)^{-\rho'+1}\left(u\,\frac{\partial f}{\partial u} + w\,\frac{\partial f}{\partial w} + bf\right)$$

and

$$t^{-\rho+1}\,(1-t)^{-\rho'}\left(v\,\frac{\partial f}{\partial v} + w\,\frac{\partial f}{\partial w} + b'f\right)$$

vanish, where

$$f = f(u, v, w)$$

$$\left[u = \frac{x}{t}, v = \frac{z}{1-t}, w = y\right]$$

satisfies the differential equations

$$[\theta_1(\theta_1+c-1) - u(\theta_1+\rho)\,(\theta_1+\psi_1+b)]\; f(u, v, w) = 0$$

$$(1\cdot4)\qquad [\phi_1(\phi_1+\psi_1+c'-1) - v(\phi_1+\rho')\,(\phi_1+\psi_1+b')]\; f(u, v, w) = 0$$

$$[\psi_1(\phi_1+\psi_1+c'-1) - w(\theta_1+\psi_1+b)\,(\phi_1+\psi_1+b')]\, f(u, v, w) = 0\,.$$

$$\theta_1, \phi_1, \psi_1, = u\frac{\partial}{\partial v}, v\frac{\partial}{\partial v}, w\frac{\partial}{\partial w}.$$

The purposes of the present paper are to study the functions contiguous to H_A and H_B, and to obtain their relations with these functions. The method employed is fully explained in the case of the function H_A. In a similar manner we can derive the contiguous function relations associated with H_B and the function H_C whose existence was recorded in [7] and [9].

In the case of Gauss's ordinary hypergeometric function [2] we know from Riemann's theorem that between

$$ {}_2F_1\,(a, b\,;\, c\,;\, z) $$

and any two hypergeometric functions contiguous to it there always exists a linear relation with polynomial coefficients. A similar remark can be made in the case of the Appell's double hypergeometric functions F_1, F_2, F_3 and F_4[1]. But when we study the contiguous function relations associated with the functions H_A, H_B and H_C it does not seem possible to make any such general statement. Nevertheless, between any contiguous function and functions of the type

$$ H(a+f,\, b+g,\, b'+h\,;\, \ldots\,;\, x, y, z) $$

where f, g, h are positive or negative integers, there does always exist a linear relation with coefficients polynomial in x, y and z. The number of contiguous functions occurring in a particular relation depends upon the individual relation and the function considered.

2. DEFINITIONS AND NOTATIONS

Following the notation of [6] the two triple hypergeometric functions are defined by the equalities [6, p. 97]

(2·1)
$$ H_A\,(a, b, b'\,;\, c, c'\,;\, x, y, z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(a)_{m+p}\,(b)_{m+n}\,(b')_{n+p}}{m!\,n!\,p!\,(c)_m\,(c')_{n+p}}\, x^m\, y^n\, z^p\,, $$

(2·2)
$$ H_B(a, b, b'\,;\, c_1, c_2, c_3\,;\, x, y, z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(a)_{m+p}\,(b)_{m+n}\,(b')_{n+p}}{m!\,n!\,p!\,(c_1)_m\,(c_2)_n\,(c_3)_p}\, x^m\, y^n\, z^p\ ; $$

and as proved in [6, § 2) the domains of (absolute) convergence for these series are given by

$$ H_A \quad : \quad r+s+t = 1+st $$

$$ H_B \quad : \quad r+s+t+2\sqrt{rst} = 1 $$

where $|\,x\,| < r$, $|\,y\,| < s$ and $|\,z\,| < t$.

We also discuss here the third hypergeometric function [9, p. 14]

$$ H_C\,(a, b, b'\,;\, c\,;\, x, y, z) $$

defined by the triple series

$$(2{\cdot}3) \qquad \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(a)_{m+p}\,(b)_{m+n}\,(b')_{n+p}}{(c)_{m+n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!},$$

which converges absolutely whenever

$$|x| < 1, \ |y| < 1 \text{ and } |z| < 1.$$

This function furnishes a new* generalization of Appell's double hypergeometric function

$$F_1(a, b, b'; c; X, Y)$$

and in a future communication we hope to incorporate a detailed discussion of its several interesting properties other than those presented in [7].

For convenience, we make use of the letters θ, ϕ and ψ to denote the operators

$$x\frac{\partial}{\partial x}, \quad y\frac{\partial}{\partial y} \text{ and } z\frac{\partial}{\partial z}$$

respectively. Also, for the sake of brevity, by $H_A(a+1)$ we shall mean

$$H_A(a+1, b, b'; c, c'; x, y, z),$$

and so on.

3. RELATIONS BETWEEN FUNCTIONS CONTIGUOUS TO H_A

From (2·1) we have

$$(3{\cdot}1) \qquad a\,H_A\,(a+1) = a\,H_A + x\frac{\partial H_A}{\partial x} + z\frac{\partial H_A}{\partial z},$$

$$(3{\cdot}2) \qquad b\,H_A\,(b+1) = b\,H_A + x\frac{\partial H_A}{\partial x} + y\frac{\partial H_A}{\partial y},$$

$$(3{\cdot}3) \qquad b'\,H_A\,(b'+1) = b'H_A + y\frac{\partial H_A}{\partial y} + z\frac{\partial H_A}{\partial z},$$

$$(3{\cdot}4) \qquad (c-1)H_A(c-1) = (c-1)H_A + x\frac{\partial H_A}{\partial x},$$

$$(3{\cdot}5) \qquad (c'-1)H_A(c'-1) = (c'-1)H_A + y\frac{\partial H_A}{\partial y} + z\frac{\partial H_A}{\partial z}$$

*For the earlier generalization of the Appell function F_1 by means of Lauricella's triple hypergeometric series

$$F_D(a, b_1, b_2, b_3; c; x, y, z)$$

see *e.g.*, [4] and [1, p. 114].

and

(3·6) $$\frac{\partial^{r+s+t} H_A}{\partial x^r\, \partial y^s\, \partial z^t} = \frac{(a)_{r+t}\,(b)_{r+s}\,(b')_{s+t}}{(c)_r\,(c')_{s+t}}$$
$$\times\, H_A(a+r+t, b+r+s, b'+s+t; c+r, c'+s+t; x, y, z).$$

Also the partial differential equations satisfied by H_A are [6, (4·1)]

(3·7)
$$(i) \qquad [\theta(\theta+c-1) - x(\theta+\psi+a)(\theta+\phi+b)]H_A = 0$$
$$(ii) \qquad [\phi(\phi+\psi+c'-1) - y(\theta+\phi+b)(\phi+\psi+b')]H_A = 0$$
$$(iii) \qquad [\psi(\phi+\psi+c'-1) - z(\theta+\psi+a)(\phi+\psi+b')]H_A = 0.$$

Now write (3·7) (i) with $a-1$ instead of a and we have
$$[\theta(\theta+c-1) - x(\theta+\psi+a-1)(\theta+\phi+b)]\, H_A(a-1) = 0,$$
or
$$[(1-x)\,\theta - x\,\phi - \psi + c - a - bx]\,(\theta+\psi+a-1)\, H_A(a-1)$$
$$+(\psi+a-1)(\psi+a-c)\, H_A(a-1) = 0.$$

By virtue of (3·1) this gives us

(3·8) $$[(1-x)\,\theta - x\,\phi - \psi + c - a - bx]\,(a-1)\, H_A$$
$$+ (\psi+a-1)(\psi+a-c)\, H_A(a-1) = 0.$$

Similarly we have

(3·9) $$[-\theta + (1-z)(\phi+\psi) + c' - a - b'z]\,(a-1)\, H_A$$
$$+ (\theta+a-1)(\theta-\phi+a-c')\, H_A(a-1) = 0.$$

(3·10) $$[(1-x)\,\theta - \phi - x\,\psi + c - b - ax]\,(b-1)\, H_A$$
$$+ (\phi+b-1)(\phi+b-c)\, H_A(b-1) = 0.$$

(3·11) $$[-\theta + (1-y)(\phi+\psi) + c' - b - b'y]\,(b-1)\, H_A$$
$$+ (\theta+b-1)(\theta-\psi+b-c')\, H_A(b-1) = 0.$$

(3·12) $$[-y\,\theta + (1-y)\,\phi + c' - b' - by]\,(b'-1)\, H_A$$
$$+ (b'-c')(\psi+b'-1)\, H_A(b'-1) = 0.$$

(3·13) $$[-z\,\theta + (1-z)\,\psi + c' - b' - az]\,(b'-1)\, H_A$$
$$+ (b'-c')(\phi+b'-1)\, H_A(b'-1) = 0.$$

(3·14) $$[(1-x)\,\theta - x\,\phi - x\,\psi - (a+b-c)x]\, c\, H_A$$
$$= x(\psi+a-c)(\phi+b-c)\, H_A(c+1).$$

(3·15) $$[-y\,\theta + (1-y)\,\phi - by]\, c'\, H_A$$
$$= y(b'-c')(\theta+\phi+b)\, H_A(c'+1).$$

(3·16) $$[-z\,\theta + (1-z)\,\psi - az]\, c'\, H_A$$
$$= z(b'-c')(\theta+\psi+a)\, H_A(c'+1).$$

By eliminating θ, ϕ and ψ between the equations (3·1)—(3·16) we can derive a number of relations connecting H_A with functions contiguous to itself and functions of the type

$$H_A(a+f, b+g, c+h; \ldots \ldots; x, y, z),$$

where f, g, h are positive or negative integers.

For instance, on using (3·3) and (3·5) we have

$$(b'-c'+1)\, H_A = b'\, H_A(b'+1) - (c'-1)\, H_A(c'-1), \tag{3·17}$$

and the relation

$$\begin{aligned}(4+a+b+b'-2c-2c')\, H_A = {} & a\, H_A(a+1) + b\, H_A(b+1) + b'\, H_A(b'+1) \\ & - 2(c-1)\, H_A(c-1) - 2(c'-1)\, H_A(c'-1)\end{aligned} \tag{3·18}$$

is obtained when we use the equations (3·1)—(3·5).

From (3·4) and (3·6) we get

$$(c-1)\, H_A = (c-1)\, H_A\,(c-1) - \left(\frac{ab}{c}\right) x\, H_A(a+1, b+1, c+1), \tag{3·19}$$

and if we combine (3·5) with (3·6), the relation

$$\begin{aligned}(c'-1)\, H_A = (c'-1)\, H_A(c'-1) & - \left(\frac{bb'}{c'}\right) y\, H_A(b+1, b'+1, c'+1) \\ & - \left(\frac{ab'}{c'}\right) z\, H_A(a+1, b'+1, c'+1)\end{aligned} \tag{3·20}$$

follows immediately.

The contiguous function relations

$$\begin{aligned}a\, H_A = a\, H_A(a+1) & - \left(\frac{ab}{c}\right) x\, H_A(a+1, b+1, c+1) \\ & - \left(\frac{ab'}{c'}\right) z\, H_A(a+1, b'+1, c'+1),\end{aligned} \tag{3·21}$$

$$\begin{aligned}b\, H_A = b\, H_A(b+1) & - \left(\frac{ab}{c}\right) x\, H_A(a+1, b+1, c+1) \\ & - \left(\frac{bb'}{c'}\right) y\, H_A(b+1, b'+1, c'+1)\end{aligned} \tag{3·22}$$

and

$$\begin{aligned}b'\, H_A = b'\, H_A(b'+1) & - \left(\frac{bb'}{c'}\right) y\, H_A(b+1, b'+1, c'+1) \\ & - \left(\frac{ab'}{c'}\right) z\, H_A(a+1, b'+1, c'+1)\end{aligned} \tag{3·23}$$

are readily obtained when we use (3·6) in (3·1), (3·2) and (3·3) respectively.

When we combine (3·2) and (3·6) with the equations (3·12) and (3·15) we get

$$(3{\cdot}24)\qquad (b'-c')\,H_A + by\,H_A(b+1) = (b'-c')\,H_A(b'-1) + \left(\frac{bb'}{c'}\right) y\,H_A(b+1,\,b'+1,\,c'+1) + \left[\frac{a(b'-c')}{c'}\right] z\,H_A(a+1,\,c'+1)$$

and

$$(3{\cdot}25)\qquad c'\,H_A = b'\,H_A(b'+1,\,c'+1) - (b'-c')\,H_A(c'+1).$$

From (3·1), (3·6) and (3·13) we obtain

$$(3{\cdot}26)\qquad (b'-c')\,H_A + a\,z\,H_A(a+1) = (b'-c')\,H_A(b'-1) + \left[\frac{b(b'-c')}{c'}\right] y\,H_A(b+1,\,c'+1) + \left(\frac{ab'}{c'}\right) z\,H_A(a+1,\,b'+1,\,c'+1).$$

Lastly, we combine (3·12) with (3·15) and then use the equations (3·2) and (3·6). We thus have

$$(3{\cdot}27)\qquad c'\,H_A = c'\,H_A(b'-1) + b\,y\,H_A(b+1,\,c'+1) + a\,z\,H_A(a+1,\,c'+1).$$

We remark that the above relation can also be obtained when we combine (3·13) with (3·16) and then apply (3·1) and (3·6).

4. CONTIGUOUS FUNCTION RELATIONS ASSOCIATED WITH H_B

From the definition of H_B we have

$$(4{\cdot}1)\qquad a\,H_B(a+1) = a\,H_B + x\frac{\partial H_B}{\partial x} + z\frac{\partial H_B}{\partial z}$$

$$(4{\cdot}2)\qquad b\,H_B(b+1) = b\,H_B + x\frac{\partial H_B}{\partial x} + y\frac{\partial H_B}{\partial y}$$

$$(4{\cdot}3)\qquad b'\,H_B(b'+1) = b'\,H_B + y\frac{\partial H_B}{\partial y} + z\frac{\partial H_B}{\partial z}$$

$$(4{\cdot}4)\qquad (c_1-1)\,H_B(c_1-1) = (c_1-1)\,H_B + x\frac{\partial H_B}{\partial x}$$

$$(4{\cdot}5)\qquad (c_2-1)\,H_B(c_2-1) = (c_2-1)\,H_B + y\frac{\partial H_B}{\partial y}$$

$$(4{\cdot}6)\qquad (c_3-1)\,H_B(c_3-1) = (c_3-1)\,H_B + z\frac{\partial H_B}{\partial z}$$

$$(4{\cdot}7)\qquad \frac{\partial^{r+s+t} H_B}{\partial x^r\,\partial y^s\,\partial z^t} = \frac{(a)_{r+t}\,(b)_{r+s}\,(b')_{s+t}}{(c_1)_r\,(c_2)_s\,(c_3)_t}$$

$$\times\ H_B(a+r+t,\ b+r+s,\ b'+s+t\,;\ c_1+r,\ c_2+s,\ c_3+t\,;\ x, y, z),$$

and the partial differential equations associated with it are [6, p. 101 (4·2)]

$$(4{\cdot}8)\qquad \begin{array}{ll} (i) & [\theta(\theta + c_1 - 1) - x(\theta + \psi + a)\,(\theta + \phi + b\,)]\ H_B = 0 \\ (ii) & [\phi(\phi + c_2 - 1) - y(\theta + \phi + b)\,(\phi + \psi + b')]\ H_B = 0 \\ (iii) & [\psi(\psi + c_3 - 1) - z(\theta + \psi + a)\,(\phi + \psi + b')]\ H_B = 0. \end{array}$$

By employing the method illustrated in the preceding section we can deduce a number of relations between functions contiguous to H_B. To quote a few, we have

$$(4{\cdot}9)\qquad (a - c_1 - c_3)\,H_B = a\,H_B(a+1) - (c_1 - 1)\,H_B(c_1 - 1) - (c_3 - 1)\,H_B(c_3 - 1)$$

$$(4{\cdot}10)\qquad (b - c_1 - c_2)\,H_B = b\,H_B(b+1) - (c_1 - 1)\,H_B(c_1 - 1) - (c_2 - 1)\,H_B(c_2 - 1)$$

$$(4{\cdot}11)\qquad (b' - c_2 - c_3)\,H_B = b'\,H_B(b'+1) - (c_2 - 1)\,H_B(c_2 - 1) - (c_3 - 1)\,H_B(c_3 - 1)$$

$$(4{\cdot}12)\qquad (c_1 - 1)\,H_B = (c_1 - 1)\,H_B(c_1 - 1) - \left(\frac{ab}{c_1}\right) x\,H_B(a+1,\ b+1,\ c_1+1)$$

$$(4{\cdot}13)\qquad (c_2 - 1)\,H_B = (c_2 - 1)\,H_B(c_2 - 1) - \left(\frac{bb'}{c_2}\right) y\,H_B(b+1,\ b'+1,\ c_2+1)$$

$$(4{\cdot}14)\qquad (c_3 - 1)\,H_B = (c_3 - 1)\,H_B(c_3 - 1) - \left(\frac{ab'}{c_3}\right) z\,H_B(a+1,\ b'+1,\ c_3+1)$$

$$(4{\cdot}15)\qquad a\,H_B = a\,H_B(a+1) - \left(\frac{ab}{c_1}\right) x\,H_B(a+1,\ b+1,\ c_1+1) - \left(\frac{ab'}{c_3}\right) z\,H_B(a+1,\ b'+1,\ c_3+1)$$

$$(4{\cdot}16)\qquad b\,H_B = b\,H_B(b+1) - \left(\frac{ab}{c_1}\right) x\,H_B(a+1,\ b+1,\ c_1+1) - \left(\frac{bb'}{c_2}\right) y\,H_B(b+1,\ b'+1,\ c_2+1)$$

$$(4{\cdot}17)\qquad b'\,H_B = b'\,H_B(b'+1) - \left(\frac{bb'}{c_2}\right) y\,H_B(b+1,\ b'+1,\ c_2+1) - \left(\frac{ab'}{c_3}\right) z\,H_B(a+1,\ b'+1,\ c_3+1).$$

5. FUNDAMENTAL EQUATIONS FOR H_C

The fundamental equations for H_C are

$$a\,H_C(a+1) = a\,H_C + x\,\frac{\partial H_C}{\partial x} + z\,\frac{\partial H_C}{\partial z} \tag{5·1}$$

$$b\,H_C(b+1) = b\,H_C + x\,\frac{\partial H_C}{\partial x} + y\,\frac{\partial H_C}{\partial y} \tag{5·2}$$

$$b'\,H_C(b'+1) = b'\,H_C + y\,\frac{\partial H_C}{\partial y} + z\,\frac{\partial H_C}{\partial z} \tag{5·3}$$

$$(c-1)\,H_C(c-1) = (c-1)\,H_C + x\,\frac{\partial H_C}{\partial x} + y\,\frac{\partial H_C}{\partial y} + z\,\frac{\partial H_C}{\partial z} \tag{5·4}$$

$$\frac{\partial^{r+s+t}\,H_C}{\partial x^r\,\partial y^s\,\partial z^t} = \frac{(a)_{r+t}\,(b)_{r+s}\,(b')_{s+t}}{(c)_{r+s+t}} \times H_C(a+r+t,\, b+r+s,\, b'+s+t;\, c+r+s+t;\, x, y, z), \tag{5·5}$$

and it satisfies the differential system [7, § 6]

$$\begin{aligned} &(i) && [\theta(\theta+\phi+\psi+c-1) - x(\theta+\psi+a)\,(\theta+\phi+b)]\,H_C = 0 \\ (5·6)\quad &(ii) && [\phi(\theta+\phi+\psi+c-1) - y(\theta+\phi+b)\,(\phi+\psi+b')]\,H_C = 0 \\ &(iii) && [\psi(\theta+\phi+\psi+c-1) - z(\theta+\psi+a)\,(\phi+\psi+b')]\,H_C = 0. \end{aligned}$$

The method of § 3 can now be employed to obtain the contiguous function relations associated with H_C.

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ON SOME INFINITE INTEGRALS—II

By

D. C. GOKHROO*

Department of Mathematics, University of Jodhpur

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ABSTRACT

In this paper we have proved a theorem on Laplace transform and two infinite integrals have been evaluated by the application of that theorem. The results proved are quite interesting and appear to be new.

1. The aim of the present note is to evaluate certain infinite integrals involving Legendre and Bessel functions with the help of 'Operational Calculus'.

As usual the conventional notation $\phi(p) \doteqdot h(t)$, will be used to denote the classical Laplace transfrom

$$\phi(p) = p \int_0^\infty e^{-pt}\, h(t)\, dt, \tag{1}$$

provided the integral is convergent and $R(p) > 0$.

2. Theorem I. If $\phi(p) \doteqdot h(t)$,

and $\psi(p,\lambda) \doteqdot K_0(\lambda t)\, h(t)$,

then

$$\int_0^\infty (a+bt+ct^2)^{-\frac{1}{2}} \left\{d+\sqrt{\frac{a+bt+ct^2}{t}}\right\}^{-1} \phi\left(d+\sqrt{\frac{a+bt+ct^2}{t}}\right) dt = \frac{2}{\sqrt{cd}}\, \psi\left(d, \sqrt{b+2\sqrt{ac}}\right), \tag{2}$$

provided that the integrals are absolutely convergent, $R(a) > 0$, $R(b) > 0$, $c > 0$, $R(d) > 0$ and $h(t)$ is independent of λ.

Proof: Since $\phi(p) = p \int_0^\infty e^{-pt}\, h(t)\, dt$,

therefore

$$\int_0^\infty (a+bt+ct^2)^{-\frac{1}{2}} \left\{d+\sqrt{\frac{a+bt+ct^2}{t}}\right\}^{-1} \phi\left\{d+\sqrt{\frac{a+bt+ct^2}{t}}\right\} dt$$

$$= \int_0^\infty (a+bt+ct^2)^{-\frac{1}{2}} \int_0^\infty e^{-\left(d+\sqrt{\frac{a+bt+ct^2}{t}}\right)x}\, h(x)\, dx\, dt$$

$$= \int_0^\infty e^{-dx}\, h(x) \int_0^\infty (a+bt+ct^2)^{-\frac{1}{2}}\, e^{-x\sqrt{\frac{a+bt+ct^2}{t}}}\, dt\, dx$$

$$= \frac{2}{\sqrt{c}} \int e^{-dx}\, K_0\left(x\sqrt{b+2\sqrt{ac}}\right) h(x)\, dx$$

$$= \frac{2}{\sqrt{c}} \frac{1}{d}\, \psi\left(d, \sqrt{b+2\sqrt{ac}}\right)$$

**Present address*: Department of Mathematics, Government College, Bhilwara (Rajasthan).

On changing the order of integration and evaluating the inner integral by means of the formula [3, p. 664 (7)]

$$\int_0^\infty (a+bt+ct^2)^{-\frac{1}{2}} e^{-2\sqrt{\frac{a+bt+ct^2}{t}}} dt = \frac{2}{\sqrt{c}} \quad K_0\left(2 \sqrt{b+2\sqrt{ac}}\right), \tag{3}$$

where $R(a) > 0$, $R(b) > 0$ and $c > 0$.

The change of order of integration can be justified by the application of de la Vallee Poussin's theorem [1, p. 504] when the integrals involved are absolutely convergent.

Example I. Taking Erdelyi [2, p. 137 (1)]

$$h(t) = t^{\mu}$$

$$\doteqdot \frac{\Gamma(\mu+1)}{p^{\mu}} = \phi(p),$$

where $R(p) > 0$ and $R(1+\mu) > 0$.
and Erdelyi, [2, p. 198 (27)]

$$\begin{aligned} K_0(\lambda t)\, h(t) &= t^{\mu} K_0(\lambda t) \\ &= \frac{\Gamma(1+\mu)\ \Gamma(1+\mu)\ \sqrt{\pi}\, p}{\sqrt{2\lambda}\ (p^2-\lambda^2)^{\frac{1}{2}(\mu+\frac{1}{2})}} \quad P_{-\frac{1}{2}}^{-\mu-\frac{1}{2}}\left(\frac{p}{\lambda}\right) \\ &= \psi(p, \lambda) \end{aligned}$$

where $R(1+\mu) > 0$ and $R(p+\lambda) > 0$.

Using the above correspondences in (2), we get

$$\int_0^\infty (a+bt+ct^2)^{-\frac{1}{2}} \left\{ d + \sqrt{\frac{a+bt+ct^2}{t}} \right\}^{-1-\mu}$$

$$= \frac{\Gamma(1+\mu)\ \sqrt{2\pi}\ \{d^2-(b+2\sqrt{ac})\}^{-\frac{1}{2}(\mu+\frac{1}{2})}}{\sqrt{c}\,(b+2\sqrt{ac})^{\frac{1}{4}}} \quad P_{-\frac{1}{2}}^{-\mu-\frac{1}{2}}\left(\frac{d}{\sqrt{b+2\sqrt{ac}}}\right) \tag{4}$$

for $R(1+\mu) > 0$, $R(d) > 0$ and $c > 0$.

Example II. Taking Erdelyi [2, p. 146 (29)]

$$\begin{aligned} h(t) &= t^{-1} e^{-\alpha/t} \\ &\doteqdot 2p\, K_0\,(2\sqrt{\alpha p}) \\ &= \phi(p), \end{aligned}$$

where $R(p) > 0$ and $R(\alpha) > 0$.
and Erdelyi, [2, p. 198 (29)]

$$\begin{aligned} K_0(\lambda t)\, h(t) &= t^{-1} e^{-\alpha/t} K_0\,(\lambda t) \\ &\doteqdot 2p\, K_0 \left[\left\{ 2\alpha\left(p + \sqrt{p^2 - \frac{\lambda}{2\alpha}}\right)\right\}^{\frac{1}{2}} \right] K_0 \left[\left\{ \lambda \left(p + \sqrt{p^2 - \frac{\lambda}{2\alpha}}\right)^{-1} \right\}^{\frac{1}{2}} \right] \\ &= \psi(p, \lambda), \end{aligned}$$

where $R(p+\lambda) > 0$ and $R(\alpha) > 0$.

Applying (2) to the above correspondences, we get

$$\int_0^\infty (a+bt+ct^2)^{-\frac{1}{2}} K_0\left[2\sqrt{\alpha\left(d+\sqrt{\frac{a+bt+ct^2}{t}}\right.}\right] dt$$

$$= \frac{2}{\sqrt{c}} K_0\left[\left\{2\alpha\left(d+\sqrt{d^2-\frac{(b+2\sqrt{ac})^{\frac{1}{2}}}{2\alpha}}\right)\right\}^{\frac{1}{2}}\right]$$

$$K_0\left[\left\{\sqrt{b+2\sqrt{ac}}\left(d+\sqrt{d^2-\frac{(b+2\sqrt{ac})^{\frac{1}{2}}}{2\alpha}}\right)^{-1}\right\}^{\frac{1}{2}}\right], \qquad (5)$$

for $R(\alpha) > 0$, $R(d) > 0$ and $c > 0$.

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ON A NEW KERNEL AND ITS RELATION WITH H-FUNCTION OF FOX

By

V. K. VARMA

Engineering College, Bilaspur, M. P.

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ABSTRACT

In this paper we have defined a function by means of a double integral and established that this function plays the role of a Fourier Kernel For particular values of the parameters the kernel function reduces to the kernel of the Hankel transform. Later on the relation between this function and the H-function of Fox has been established.

1. Let $f(x)$, $g(x)$ and $K(x)$ be three functions satisfying the integral equation

(1·1) $$f(x) = \int_0^\infty K(xy)\, g(y)\, dy$$

then $K(x)$ is said to be a symmetrical fourier kernel if the integral equation holds good when $f(x)$ and $g(x)$ are interchanged amongst themselves.

A function $R(s)$ is said to be the Mallin transform of $K(x)$ if

(1·2) $$R(s) = \int_0^\infty K(x)\, x^{s-1}\, dx.$$

In a recent paper[1] Fox has defined a symmetrical Fourier Kernel $H(x)$ in the form

(1·3) $$H(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\prod_{i=1}^{q} \Gamma(b_i + c_i s) \prod_{i=1}^{p} \Gamma(a_i - e_i s)\, x^{-s}}{\prod_{i=1}^{q} \Gamma(b_i + c_i - c_i s) \prod_{i=1}^{p} \Gamma(a_i - e_i + e_i s)}\, ds,$$

where $p > q + 1$, $\sigma > 0$, $b_i > 0$, $c_i > 0$, $i = 1, \ldots, q$
and $a_i > 0$, $e_i > 0$ $i = 1, \ldots, p$.

We shall denote it symbolically by

(1·4) $$H(x) = H_{q,p}\,(x \mid b_i + c_i\, s,\ a_j - e_j s)$$

The object of this paper is to establish the kernel property of a function, defined as a double integral involving products of Bessel functions and to establish its relation with a kernel of Fox.

The new Kernel reduces to the kernel of the Hankel transform for particular values of parameters. In a recent paper[3(i)], I have proved that the function

(1·5) $$\psi_{\nu,\lambda}(x) = \frac{1}{\mu} \int_0^\infty \sqrt{x/y}\, J_\nu(x/y)\, y^{1/\mu - 3/2}\, J_\lambda(y^{\frac{1}{\mu}})\, dy$$

plays the role of a kernel of a transform for $\mu > 0$, $\nu + \frac{1}{2} > 0$, $\lambda + \frac{1}{2} > 0$, where $J_\nu(x)$ is the Bessel function of order ν.

For $\mu = 1$ the kernel in (1·5) reduces to known kernel $\omega_{\nu,\lambda}(x)$ Watson[4]. A generalisation of the Kernel $\psi_{\nu,\lambda}(x)$ has recently been given by me[3(iii)]. Here another generalisation will be given.

We have also proved[3(ii)] that

(1·6) $\qquad \psi_{\nu,\lambda}(x) \sim O(x^{-\frac{1}{4}})$ when $x \to \infty$ and is

(1·7) $\qquad O(x^{\nu+\frac{1}{2}} + x^{(\lambda+1)/\mu-\frac{1}{2}})$ for small x.

2. I define the new kernel by the integral relation

$$\psi^{\xi}_{\nu,\lambda}(x) = \frac{1}{\mu'}\int_0^\infty \psi_{\nu,\lambda}(xy)\, y^{1/\mu'-\frac{1}{2}}\, J_{\xi}\,(y^{1/\mu'})\, dy$$

$$= \frac{\sqrt{x}}{\mu\,\mu'}\int_0^\infty\int_0^\infty J_\nu\left(\frac{xy}{z}\right) J_{\xi}\,(y^{1/\mu'}) J_\lambda(z^{1/\mu'})\, z^{1/\mu-2}\, y^{1/\mu'}\, dz\, dy$$

where it is not possible to permute ν, λ, ξ amongst themselves and μ, μ' are quite independent of each other. We take $\mu > 0$, $\mu' > 0$, $\nu + \frac{1}{2} > 0$, $\lambda + \frac{1}{2} > 0$, $\xi + \frac{1}{2} > 0$.

3. First of all we shall prove that $\psi^{\xi}_{\nu,\lambda}(x)$ is bounded for all values of x. We take $0 < \mu' < 1$.

Now by virtue of (1·6) it follows that for $x > 0$

$$\frac{1}{\mu'}\int_{\gamma'}^{\gamma''} \psi_{\nu,\lambda}(xy)\, y^{1/\mu'-\frac{1}{2}}\, J_{\xi}\,(y^{1/\mu'})\, dy$$

$$\sim \frac{x^{-\frac{1}{4}}}{\mu'}\int_{\gamma}^{\gamma'} y^{1/\mu'-3/4}\; J_{\xi}\,(y^{1/\mu'})\, dy = x^{-\frac{1}{4}}\int_{\gamma_1}^{\gamma_2} y^{\mu'/4}\, J_{\xi}\,(y)\, dy$$

Then provided $\mu' < 2$, there exists a number γ_0 such that if

$$\gamma_2 > \gamma_1 > \gamma_0,\ \left|\int_{\gamma_1}^{\gamma_2} y^{\mu'/4}\, J_{\xi}\,(y)\, dy\, x^{-\frac{1}{4}}\right| < \varepsilon$$

the same γ_0 serving for all $x > 0$.

Consequently

$$\int_a^\infty \psi_{\nu,\lambda}(xy)\, y^{1/\mu'-\frac{1}{2}}\, J_{\xi}\,(y^{1/\mu'})\, dy$$

is uniformly convergent for $x > 0$ *i.e.* is bounded for finite x and also as $x \to \infty$.

Further from (1·6) and (1·7) we note that if $\nu + \frac{1}{2} > 0$, $(\lambda + 1)/\mu > \frac{1}{2}$, $\psi_{\nu,\lambda}(xy)$ is bounded for all values of x, y.

Consequently

$$\left|\int_0^a \psi_{\nu,\lambda}(xy)\, y^{1/\mu'-\frac{1}{2}}\, J_{\xi}\,(y^{1/\mu'})\, dy\right|$$

$$\leqslant |\,\psi_{\nu,\lambda}(xy)\,| \int_0^a |\,y^{1/\mu'-\frac{1}{2}}\, J_{\xi}\,(y^{1/\mu'})\,|\, dy \leqslant k\int_0^{a'} |\,y^{\mu'/2} J_{\xi}\,(y)\, dy$$

Since $\xi + \mu'/2 + 1 > 0$, the integral converges. Thus $\psi^{\xi}_{\nu,\lambda}(xy)$ is bounded for $0 < x \leqslant \alpha$, where α may be arbitrary.

4. To obtain the Mellin Barne's type integral representation for $\psi^{\xi}_{\nu,\lambda}(x)$ we apply theorem 43 [2, p. 60]. We shall discuss the case $0 < \mu' < 1$. The case $\mu' > 1$ is easy to prove.

(i) Let $K(x) = \psi_{\nu,\lambda}(x)$ and $l(x) = \frac{1}{\mu'} x^{1/\mu'-\frac{1}{2}} J_{\xi}(x^{1/\mu'})$ then $K(x)$ and $l(x)$ are integrable over any finite interval not ending at $x = 0$

(ii) From [3(i)] we have

$$F(s) = \int_0^\infty \psi_{\nu,\lambda}(x)\, x^{s-1}\, dx \tag{4·1}$$

$$= \frac{2^{(s-\frac{1}{2})(1+\mu)}\, \Gamma\left(\frac{s+\nu+\frac{1}{2}}{2}\right) \Gamma\left(\frac{\mu s+\lambda+1-\mu/2}{2}\right)}{\Gamma\left(\frac{\nu-s+\frac{3}{2}}{2}\right) \Gamma\left(\frac{\lambda-\mu s+1+\mu/2}{2}\right)}$$

which is valid for $-\nu-\frac{1}{2}$, $-(\lambda+1)/\mu+\frac{1}{2} < \sigma = -\delta < \frac{1}{4}$ where δ is a positive number.

(iii) Also we have

$$G(s) = \frac{2^{\mu'(s-\frac{1}{2})}\, \Gamma(\xi/2+\mu' s/2-\mu'/4+\frac{1}{2})}{\Gamma(\xi/2-\mu' s/2+\mu'/4+\frac{1}{2})} \tag{4·2}$$

$$= \frac{1}{\mu'} \int_0^\infty x^{1/\mu'-\frac{1}{2}} J_{\xi}(x^{1/\mu'})\, x^{s-1}\, dx = \int_0^\infty J_{\xi}(x)\, x^{-\mu'/2+\mu' s}\, dx$$

The result being valid for $\frac{1}{2}-(\xi+1)/\mu' < \sigma' = 1+\delta < \frac{1}{2}+\frac{1}{2}\mu'$.

Since $1/\mu' > 1$, it is possible to choose a positive δ to satisfy the inequalities in (ii) and (iii).

(iv) $e^{-c|s|}F(s)$ and $e^{-c|s|}G(s)$ are bounded for some positive c,

(v)
$$\frac{\eta^{1+\delta}}{\eta^{c}+\eta^{-c}} \int_0^\infty K(x\eta)\, l(x)\, dx$$

is bounded and converges in the neighbourhood of $\eta = 1$ vide article 3.

Hence by the theorem referred to above we have

$$\psi^{\xi}_{\nu,\lambda}(x) = \int_0^\infty K(xy)\, l(y)\, dy = \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} F(s)\, G(1-s)\, x^{-s}\, ds \tag{4·3}$$

where

$$F(s)\, G(1-s) = \frac{2^{\theta}\, \Gamma\left(\frac{s+\nu+\frac{1}{2}}{2}\right) \Gamma\left(\frac{\lambda+\mu s-\mu/2+1}{2}\right) \Gamma\left(\frac{\xi-\mu' s+\mu'/2+1}{2}\right)}{\Gamma\left(\frac{\nu-s+\frac{3}{2}}{2}\right) \Gamma\left(\frac{\lambda-\mu s+\mu/2+1}{2}\right) \Gamma\left(\frac{\xi+\mu' s-\mu'/2+1}{2}\right)}$$

$$= \phi(s) \text{ say,}$$

where $\theta = (s-\frac{1}{2})(1+\mu-\mu')$.

From the asymptotic behaviour of $\Gamma(A+it)$ as $t \to \infty$, it follows that the integral on the right of (4·3) converges if $c < (1-\mu+\mu')/(1+\mu-\mu')$.

Let $-\nu-\frac{1}{2}$, $-(\lambda+1)/\mu+\frac{1}{2} < Re(s) < (\xi+1)/\mu'+\frac{1}{2}$ to ensure that the increasing and decreasing sequences of poles of $\phi(s)$ do not overlap, then by evaluating the r.h. integral of (4·3) by calculus of residues, we get

(4·4) $$\psi^{\xi}_{\nu,\lambda}(x)$$

$$= \frac{x^{\nu+\frac{1}{2}}}{2^k} \sum_{n=0}^{\infty} \frac{(-1)^n\, \Gamma\left(\frac{\lambda-\mu-\mu\nu+1}{2}-\mu n\right) \Gamma\left(\frac{\xi+\mu'\nu+\mu'+1}{2}+\mu' n\right)\left(\frac{x^2}{4^{1+\mu-\mu'}}\right)^n}{n!\,\Gamma(\nu+n+1)\,\Gamma\left(\frac{\lambda+\mu\nu+\mu+1}{2}+\mu n\right)\Gamma\left(\frac{\xi-\mu'\nu-\mu'+1}{2}-\mu' n\right)}$$

$$+ \frac{x^{(\lambda+1)/\mu-\frac{1}{2}}}{2^{k'}} \sum_{n=0}^{\infty} \frac{(-1)^n\, \Gamma\left(\frac{\nu+1}{2}-\frac{\lambda+1+2n}{2\mu}\right) \Gamma\left(\frac{\xi+1}{2}+\frac{(\lambda+1+2n)\mu'}{2\mu}\right)\left(\frac{x^2}{4^{1+\mu-\mu'}}\right)^{n/\mu}}{n!\,\Gamma(\lambda+1+n)\,\Gamma\left(\frac{\nu+1}{2}+\frac{\lambda+1+2n}{2\mu}\right)\Gamma\left(\frac{\xi+1}{2}-\frac{(\lambda+1+2n)\mu'}{2\mu}\right)}$$

where $k = (\nu+1)(1+\mu-\mu')-1$, $k' = (\lambda+1)(1+\mu-\mu')/\mu - 1$.

From (4·4) it follows that for small x.

(4·5) $$\psi^{\xi}_{\nu,\lambda}(x) \sim 0\,(x^{\nu+\frac{1}{2}} + x^{(\lambda+1)/\mu-\frac{1}{2}}).$$

If we put $\mu = \mu' \neq 0$, $\lambda = \xi$, the second series on left of (4·4) vanishes, while the first gives

(4·6) $$\left\{\psi^{\xi}_{\nu,\lambda}(x)\right\}_{\substack{\mu=\mu'\\ \lambda=\xi}} = \frac{x^{\nu+\frac{1}{2}}}{2^{\nu}} \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n!\,\Gamma(\nu+n+1)} = \sqrt{x}\, J_\nu(x).$$

Moreover it can be seen that $\phi(s)$ satisfies

(4·7) $$\phi(s)\,\phi(1-s) = 1,$$

which is a formal condition for $\psi^{\xi}_{\nu,\lambda}(x)$ to play the role of a kernel of a transform, so that if $f(x)$ and $g(x)$ satisfy

(4·8) $$g(x) = \int_0^\infty \psi^{\xi}_{\nu,\lambda}(xy)\, f(y)\, dy;$$

we expect that

(4·9) $$f(x) = \int_0^\infty \psi^{\xi}_{\nu,\lambda}(xy)\, g(y)\, dy$$

will also be satisfied.

5. To establish the inversion formula in (4·9) we shall use the theory of convergence in mean square sence.

We have earlier shown that

$$(5\cdot1) \qquad \phi(\tfrac{1}{2} + i\,t)\,\phi(\tfrac{1}{2} - i\,t) = 1$$

Let $\psi(x)$ stand for $\psi^{\xi}_{\nu,\lambda}(x)$ for brevity then we have shown in (4·3) that the formula

$$(5\cdot2) \qquad \psi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(c + i\,t)\, x^{-c-i\,t}\, dt$$

is valid for some c. But we have to extend this to the case when $c = \frac{1}{2}$, before we can establish the relation of $\psi(x)$ with an H–function. Let

$$(5\cdot3) \qquad \psi_1(x) = \int_0^x \psi(u)\, du$$

then by formal integration of (5·2) we get on writing $c = \frac{1}{2}$

$$(5\cdot4) \qquad \psi_1(x) = \frac{x}{2\pi} \underset{T\to\infty}{l.\,i.\,m.} \int_{-T}^{T} \left\{\frac{\phi(\frac{1}{2} + i\,t)}{\frac{1}{2} - i\,t}\right\} x^{-\frac{1}{2}-i\,t}\, dt,$$

where $l.\ i.\ m.$ denotes limit in the mean square sence.

Since $|\,\phi(\frac{1}{2} + i\,t)\,| = 1$, $\phi(\frac{1}{2} + i\,t)/(\frac{1}{2} - i\,t) \;\varepsilon\; L^2(-\infty, \infty)$.

Further if $f(x)$ be any function ε $L^2(0, \infty)$, the conditions of theorem 129 [2, p. 221] are satisfied.

Hence by the theorem

$$(5\cdot5) \qquad g(x) = \frac{d}{dx}\int_0^{\infty} \psi_1(xu)\, f(u)\, \frac{du}{u}$$

defines almost everywhere a function ε $L^2(0, \infty)$ and the reciprocal formula

$$(5\cdot6) \qquad f(x) = \frac{d}{dx}\int_0^{\infty} \psi_1(xu)\, g(u)\frac{du}{u}$$

also holds good almost everywhere.

The above transformations are expressed in terms of $\psi_1(x)$ which may not necessarily be differentiable. In order to obtain the forms in (4·8), (4·9) we require two more conditions of theorem 132 [2, p. 227] over and above the conditions (5·1)–(5·4), which cover the first condition of theorem 132.

The other condition to be satisfied is that $\psi_1(x)$ is the integral of $\psi(x)$. For this we observe that on $Re(s) = \frac{1}{2}$, $\{\phi(s)/(1-s)\}\, x^{-s} \;\varepsilon\; L^2\,(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. Hence the integral can be evaluated by mean square methods by enclosing the poles to the left by a large semi circle with $Re(s) = \frac{1}{2}$ as the bounding diameter, and then the summing up of the residues at the poles gives the individual powers of $\psi(x)$ integrated between 0 to x. Since $\psi(x)$ is an entire function vide article 3 and (4·5) the term by term integration is justifiable and the value of this integral will be $\psi_1(x)$ which is thus an entire function. The third condition that $x^{-\frac{1}{2}}\ \psi_1(x)$ should be bounded is therefore satisfied.

Lastly if we assume that

$$(5\cdot7) \qquad f(x) = \frac{1}{x}\int_0^x \phi(y)\, dy$$

where $\phi(y) \varepsilon L^2(0, \infty)$, all conditions of theorem 132 are satisfied so that

$$f(x) = \int_0^\infty \psi(xu)\, du \quad \int_0^\infty \psi(uy)\, f(y)\, dy \tag{5·8}$$

holds for every positive x.

6. We have thus established that $\psi^{\xi}_{\nu,\lambda}(x)$ plays the role of a kernel of a transform. We can now compare the r. h. sides of (1·3) and (4·3) to give another representation for an H-function in the form

$$\begin{aligned} &2^{-h/2}\, H_{2,1}\,(2^{-h}\, x/b_1 + c_1 s,\ b_2 + c_2 s\ ;\ a_1 - e_1\, s) \\ &= \psi^{\xi}_{\nu,\lambda}(x) \\ &= \frac{\sqrt{x}}{\mu\,\mu'} \int_0^\infty \int_0^\infty J_\nu(xy/z) J_\lambda(z^{1/\mu}) J_\xi\,(y^{1/\mu'})\, z^{1/\mu - 2}\, y^{1/\mu'}\, dz\, dy \end{aligned} \tag{6·1}$$

where $b_1 = \nu/2 + \frac{1}{4}$, $b_2 = \lambda/2 - \mu/4 + \frac{1}{2}$, $a_1 = \xi/2 + \mu'/4 + \frac{1}{2}$, $c_1 = \frac{1}{2}$, $c_2 = \mu/2$, $e_1 = \mu'/2$, $h = 1 + \mu - \mu'$.

If we apply Parseval's theorem [2, p. 52] in the form

$$\frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} F(s)\, G(s)\, x^{-s}\, ds = \int_0^\infty g(u)\, f(x/u)\, du/u$$

with $\quad g(u) = \sqrt{u}\, J_\nu(u)$ and $f(u) = H_{1,1}(u/b + c\, s\ ;\ a - e\, s)$

where $\quad b = (\lambda - \mu/2 + 1)/2,\ c = \mu/2,\ a = (\xi + \mu'/2 + 1)/2,\ e = \mu'/2,$

we get with the help of (1·3) and (4·3) another relation in the form

$$\psi^{\xi}_{\nu,\lambda}(x) = 2^{-h} \int_0^\infty H_{1,1}\,(2^{-h}\, x/y/b + c\, s\ ;\ a - e\, s)\, y^{-\frac{1}{2}}\, J_\nu(y)\, dy \tag{6·2}$$

showing that $\psi^{\xi}_{\nu,\lambda}(x)$ and $2^{-h}\, H_{1\,1}\,(2^{-h}\, x^{-1}/b + c\, s\ ;\ a - e\, s)/x$ are Hankel transforms of each other of order ν.

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THE CONFLUENT HYPERGEOMETRIC FUNCTIONS OF THREE VARIABLES

By

R. N. JAIN

Holkar Science College, Indore, M. P.

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ABSTRACT

In this paper thirty-eight confluent hypergeometric functions of three variables have been studied. The confluent functions are the limiting cases of fourteen hypergeometric functions of three variables discovered by G. Lauricella and S. Saran. The relations discussed are analogous to those discussed by P. Humbert for confluent functions of two variables.

1. INTRODUCTION

This paper is devoted to the study of certain new functions which may be regarded as limiting cases of the 'hypergeometric functions of three variables' discovered by Lauricella [1 ; Chap. VII] and Saran [7]. The relation which the new functions bear to the Lauricella's and Saran's functions is, in fact, analogous to that which the confluent hypergeometric functions of two variables Φ_1, Φ_2, Φ_3 Ψ_1, Ψ_2, Ξ_1 ane Ξ_2 [5] bear to Appell's hypergeometric functions of two variables. These functions satisfy partial differential equations and can be expressed as definite integrals.

There are fourteen hypergeometric functions of three variables. Four of these functions, *i.e.* F_A, F_B, F_C, F_D were discussed by Lauricella [1] who conjectured the existence of ten other such fuuctions. Later on the Lauricella's series of functions of three variables was completed by Saran [7] who introduced the ten functions F_E, F_F, F_G, F_K, F_M, F_N, F_P, F_R, F_S and F_T.

We use the usual notation :

$$(a, n) \equiv a\,(a+1)\ \ldots\ (a+n-1);\ (a,o) \equiv 1.$$

2. FORMATION OF THE CONFLUENT HYPERGEOMETRIC SERIES

The confluent hpyergeometric functions of three variables may be formed by confluence from Lauricella's and Saran's functions in the following way :—

In Laurícella's function $F_A\,(a, b_1, b_2, b_3 ; c_1, c_2, c_3 ; x, y, z)$, change z to z/b_3 and let $b_3 \to \infty$. We thus obtain the first of our confluent functions :

(2.1) $$_3\Phi_A^{(1)}\,(a, b_1, b_2 ; c_1, c_2, c_3 ; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a, m+n+p)\,(b_1, m)\,(b_2, n)}{(1, m)\,(1, n)\,(1, p)\,(c_1, m)\,(c_2, n)\,(c_3, m)}\, x^m\, y^n\, z^p.$$

A second function can be obtained from F_A by changing y to y/b_3, z to z/b_3 and making $b_2 \to \infty$, $b_3 \to \infty$.

We thus obtain the function

$$(2{\cdot}2) \qquad {}_3\Phi_A^{(2)}(a, b_1 ; c_1, c_2, c_3 ; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a, m+n+p)\,(b_1, m)}{(1, m)\,(1, n)\,(1, p)\,(c_1, m)\,(c_2, n)\,(c_3, p)}\, x^m\, y^n\, z^p.$$

A third new function can be derived from F_A by changing x to x/b_1, y to y/b_2, z to z/b_3 and making $b_1 \to \infty$, $b_2 \to \infty$, $b_3 \to \infty$. We are thus led to the function

$$(2{\cdot}3) \qquad {}_3\Phi_A^{(3)}(a ; c_1, c_2, c_3 ; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a, m+n+p)}{(1, m)\,(1, n)\,(1, p)\,(c_1, m)\,(c_2, n)\,(c_3, p)}\, x^m\, y^n\, z^p.$$

Taking next the remaining thirteen functions of Lauricella and Saran, and applying to them a similar process, we obtain thirty-five more functions :

$$(2{\cdot}4) \qquad {}_3\Phi_B^{(1)}(a_1, a_2, a_3, b_1, b_2 ; c ; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1, m)\,(a_2, n)\,(a_3, p)\,(b_1, m)\,(b_2, n)}{(1, m)\,(1, n)\,(1, p)\,(c, m+n+p)}\, x^m\, y^n\, z^p.$$

$$(2{\cdot}5) \qquad {}_3\Phi_B^{(2)}(a_1, a_2, a_3 ; b_1 ; c ; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1, m)\,(a_2, n)\,(a_3, p)\,(b_1, m)}{(1, m)\,(1, n)\,(1, p)\,(c, m+n+p)}\, x^m\, y^n\, z^p.$$

$$(2{\cdot}6) \qquad {}_3\Phi_B^{(3)}(a_1, a_2, a_3 ; c ; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1, m)\,(a_2, n)\,(a_3, p)}{(1, m)\,(1, n)\,(1, p)\,(c, m+n+p)}\, x^m\, y^n\, z^p.$$

$$(2{\cdot}7) \qquad {}_3\Phi_B^{(4)}(a_1, a_1 ; c ; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1, m)\,(a_2, n)}{(1, m)\,(1, n)\,(1, p)\,(c, m+n+p)}\, x^m\, y^n\, z^p.$$

$$(2{\cdot}8) \qquad {}_3\Phi_B^{(5)}(a_1, a_2, b_1, b_2 ; c ; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1, m)\,(a_2, n)\,(b_1, m)\,(b_2, n)}{(1, m)\,(1, n)\,(1, p)\,(c, m+n+p)}\, x^m\, y^n\, z^p.$$

$$(2{\cdot}9) \qquad {}_3\Phi_D^{(1)}(a, b_1, b_2 ; c ; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a, m+n+p)\,(b_1, m)\,(b_2, n)}{(1, m)\,(1, n)\,(1, p)\,(c, m+n+p)}\, x^m\, y^n\, z^p.$$

$$(2\cdot10)\qquad {}_3\Phi_E^{(1)}(a_1, a_1, a_1, b_2, b_2\,;\, c_1\, c_2, c_3\,;\, x, y, z)$$
$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1, m+n+p)\,(b_2, n+p)}{(1, m)\,(1, n)\,(1, p)\,(c_1, m)\,(c_2, n)\,(c_3, p)}\, x^m y^n z^p.$$

$$(2\cdot11)\qquad {}_3\Phi_F^{(1)}(a_1, a_1, a_1, b_1, b_1\,;\, c_1, c_1, c_2;\, x, y, z)$$
$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1, m+n+p)\,(b_1, m+p)}{(1, m)\,(1, n)\,(1, p)\,(c_1, m)\,(c_2, n+p)}\, x^m y^n z^p.$$

$$(2\cdot12)\qquad {}_3\Phi_F^{(2)}(a_1, a_1, a_1, b_2\,;\, c_1, c_2, c_2\,;\, x, y, z)$$
$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1\; m+n+p)\,(b_2, n)}{(1, m)\,(1, n)\,(1, p)\,(c_1, m)\,(c_2, n+p)}\, x^m y^n z^p$$

$$(2\cdot13)\qquad \Phi_F^{(3)}(b_1, b_2, b_1\,;\, c_1\, c_2, c_2\,;\, x, y, z)$$
$$= \sum_{m,n,p=0}^{\infty} \frac{(b_1, m+p)\,(b_2, n)}{(1, m)\,(1, n)\,(1, p)\,(c_1, m)\,(c_2, n+p)}\, x^m y^n z^p.$$

$$(2\cdot14)\qquad {}_3\Phi_F^{(4)}(b_1, b_1;\, c_1, c_2, c_2\,;\, x, y, z)$$
$$= \sum_{m,n,p=0}^{\infty} \frac{(b_1, m+p)}{(1, m)\,(1, n)\,(1, p)\,(c_1, m)\,(c_2, n+p)}\, x^m y^n z^p.$$

$$(2\cdot15)\qquad {}_3\Phi_F^{(5)}(b_2\,;\, c_1, c_2, c_2\,;\, x, y, z)$$
$$= \sum_{m,n,p=0}^{\infty} \frac{(b_2, n)}{(1, m)\,(1, n)\,(1, p)\,(c_1, m)\,(c_2, n+p)}\, x^m y^n z^p.$$

$$(2\cdot16)\qquad {}_3\Phi_G^{(1)}(a_1, a_1, a_1, b_1, b_2\,;\, c_1, c_2, c_2\,;\, x, y, z)$$
$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1, m+n+p)\,(b_1, m)\,(b_2, n)}{(1, m)\,(1, n)\,(1, p)\,(c_1, m)\,(c_2, n+p)}\, x^m y^n z^p.$$

$$(2\cdot17)\qquad {}_3\Phi_K^{(1)}(a_1, a_2, a_2, b_1\, b_1\,;\, c_1, c_2, c_3\,;\, x, y, z)$$
$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1, m)\,(a_2, n+p)\,(b_1, m+p)}{(1, m)\,(1, n)\,(1, p)\,(c_1, m)\,(c_1, n)\,(c_2, p)}\, x^m y^n z^p.$$

$$(2\cdot18)\qquad {}_3\Phi_K^{(2)}(a_2, a_2, b_1, b_1\,;\, c_1, c_2, c_3\,;\, x, y, z)$$
$$= \sum_{m,n,p=0}^{\infty} \frac{(a_2, n+p)\,(b_1, m+p)}{(1, m)\,(1, n)\,(1, p)\,(c_1, m)\,(c_2, n)\,(c_3, p)}\, x^m y^n z^p.$$

$$(2\cdot19)\qquad {}_3\Phi_M^{(1)}(a_1, a_2, a_2, b_1, b_2\,;\, c_1, c_2, c_2\,;\, x, y, z)$$
$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1, m)\,(a_2, n+p)\,(b_1, m+p)}{(1, m)\,(1, n)\,(1, p)\,(c_1, m)\,(c_2, n+p)}\, x^m y^n z^p.$$

(2·20) $${}_3\Phi_M^{(2)}\ (a_2, b_1, b_2, b_1\,;\ c_1, c_2, c_2\,;\ x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1, m)\ (b_1, m+p)\ (b_2, n)}{(1, m)\ (1, n)\ (1, p)\ (c_1, m)\ (c_2, n+p)}\ x^m\, y^n\, z^p.$$

(2·21) $${}_3\Phi_M^{(3)}\ (a_2, a_2, b_1, b_1\,;\ c_1, c_2, c_2\,;\ x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_2, n+p)\ (b_1, m+p)}{(1, m)\ (1, n)\ (1, p)\ (c_1, m)\ (c_2, n+p)}\ x^m\, y^n\, z^p.$$

(2·22) $${}_3\Phi_M^{(4)}\ (a_1, b_1, b_1;\ c_1, c_2, c_2\,;\ x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1, m)\ (b_1, m+p)}{(1, m)\ (1, n)\ (1, p)\ (c_1, m)\ (c_2, n+p)}\ x^m\, y^n\, z^p.$$

(2·23) $${}_3\Phi_N^{(1)}\ (a_1, a_2, a_3, b_1, b_1\,;\ c_1, c_2, c_2\,;\ x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1, m)\ (a_2, n)\ (a_3, p)\ (b_1, m+p)}{(1, m)\ (1, n)\ (1, p)\ (c_1, m)\ (c_2, n+p)}\ x^m\, y^n\, z^p.$$

(2·24) $${}_3\Phi_N^{(2)}\ (a_1, a_2, b_1, b_2, b_1;\ c_1, c_2, c_2\,;\ x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1, m)\ (a_2, n)\ (b_1, m+p)\ (b_2, n)}{(1, m)\ (1, n)\ (1, p)\ (c_1, m)\ (c_2, n+p)}\ x^m\, y^n\, z^p.$$

(2·25) $${}_3\Phi_N^{(3)}\ (a_2, b_1, b_2, b_1;\ c_1, c_2, c_2;\ x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_2, n)\ (b_1, m+p)\ (b_2, n)}{(1, m)\ (1, n)\ (1, p)\ (c_1, m)\ (c_2, n+p)}\ x^m\, y^n\, z^p.$$

(2·26) $${}_3\Phi_P^{(1)}\ (a_1, a_2, a_1, b_1, b_1;\ c_1, c_2, c_2;\ x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1, m+p)\ (a_2, n)\ (b_1, m+n)}{(1, m)\ (1, n)\ (1, p)\ (c_1, m)\ (c_2, n+p)}\ x^m\, y^n\, z^p.$$

(2·27) $${}_3\Phi_P^{(2)}\ (a_1, a_2, a_1, b_2;\ c_1, c_2, c_2\,;\ x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1, m+p)\ (a_2, n)\ (b_2, p)}{(1, m)\ (1, n)\ (1, p)\ (c_1, m)\ (c_2, n+p)}\ x^m\, y^n\, z^p.$$

(2·28) $${}_3\Phi_P^{(3)}\ (a_1, a_1, b_1, b_1\,;\ c_1, c_2, c_2\,;\ x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1, m+p)\ (b_1, m+n)}{(1, m)\ (1, n)\ (1, p)\ (c_1, m)\ (c_2, n+p)}\ x^m\, y^n\, z^p$$

(2·29) $${}_3\Phi_R^{(1)}\ (a_1, a_2, a_1, b_1, b_1\,;\ c_1, c_2, c_2\,;\ x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1, m+p)\ (a_2, n)\ (b_1, m+p)}{(1, m)\ (1, n)\ (1, p)\ (c_1, m)\ (c_2, n+p)}\ x^m\, y^n\, z^p.$$

(2·30) $$ {}_3\Phi\,{}^{(2)}_{R}\,(a_1, a_1, b_1, b_1;\, c_1, c_2, c_2;\, x, y, z) $$

$$ = \sum_{m,n,p=0}^{\infty} \frac{(a_1, m+p)\,(b_1, m+p)}{(l, m)\,(l, n)\,(l, p)\,(c_1, m)\,(c_2, n+p)}\, x^m y^n z^p. $$

(2·31) $$ {}_3\Phi\,{}^{(1)}_{S}\,(a_1, a_2, a_2, b_1, b_2;\, c_1, c_1, c_1;\, x, y, z) $$

$$ = \sum_{m,n,p=0}^{\infty} \frac{(a_1, m)\,(a_2, n+p)\,(b_1, m)\,(b_2, n)}{(1, m)\,(1, n)\,(1, p)\,(c_1, m+n+p)}\, x^m y^n z^p. $$

(2·32) $$ {}_3\Phi\,{}^{(2)}_{S}\,(a_1, a_2, a_2, b_2, b_3;\, c_1, c_1, c_1;\, x, y, z) $$

$$ = \sum_{m,n,p=0}^{\infty} \frac{(a_1, m)\,(a_2, n+p)\,(b_2, n)\,(b_3, p)}{(1, m)\,(1, n)\,(,1\, p)\,(c_1, m+n+p)}\, x^m y^n z^p. $$

(2·33) $$ {}_3\Phi\,{}^{(3)}_{S}\,(a_1, a_2, a_2, b_1;\, c_1, c_1, c_1;\, x, y, z) $$

$$ = \sum_{m,n,p=0}^{\infty} \frac{(a_1, m)\,(a_2, n+p)\,(b_1, m)}{(1, m)\,(1, n)\,(1, p)\,(c_1, m+n+p)}\, x^m y^n z^p. $$

(2·34) $$ {}_3\Phi\,{}^{(4)}_{S}\,(a_2, a_2, b_2, b_3;\, c_1, c_1, c_1;\, x, y, z) $$

$$ = \sum_{m,n,p=0}^{\infty} \frac{(a_2, n+p)\,(b_2, n)\,(b_3, p)}{(1, m)\,(1, n)\,(1, p)\,(c_1, m+n+p)}\, x^m y^n z^p. $$

(2·35) $$ {}_3\Phi\,{}^{(5)}_{S}\,(a_2, a_2, b_2;\, c_1, c_1, c_1, x, y, z) $$

$$ = \sum_{w,n,p=0}^{\infty} \frac{(a_2, n+p)\,(b_2, n)}{(1, m)\,(1, n)\,(1, p)\,(c_1, m+n+p)}\, x^m y^n z^p. $$

(2·36) $$ {}_4\Phi\,{}^{(1)}_{T}\,(a_1, a_2, a_2, b_1, b_1;\, c_1, c_1, c_1;\, x, y, z) $$

$$ = \sum_{m,n,p=0}^{\infty} \frac{(a_1, m)\,(a_2, n+p)\,(b_1, m+p)}{(1, m)\,(1, n)\,(1, p)\,(c_1, m+n+p)}\, x^m y^n z^p. $$

(2·37) $$ {}_3\Phi\,{}^{(2)}_{T}\,(a_1, a_2, a_2, b_2;\, c_1, c_1, c_1;\, x, y, z) $$

$$ = \sum_{m,n,p=0}^{\infty} \frac{(a_1, m)\, a_2, n+p)\,(b_2, n)}{(1, m)\,(1, n)\,(1, p)\,(c_1, m+n+p)}\, x^m y^n z^p. $$

(2·38) $$ {}_3\Phi\,{}^{(3)}_{T}\,(a_2, a_2, b_1, b_1;\, c_1, c_1, c_1;\, x, y, z) $$

$$ = \sum_{m,n,p=0}^{\infty} \frac{(a_2, n+p)\,(b_1, m+p)}{(1, m)\,(1, n)\,(1, p)\,(c_1, m+n+p)}\, x^m y^n z^p. $$

Out of the thiry-eight functions which we have discussed above, only three have been mentioned previously. The function ${}_3\Phi\,{}^{(3)}_{B}$ occurs in [3] as ${}_3\Phi\,(a_1, a_2, a_3;\, c;\, x, y, z)$ and in [4 ; p. 445] as $\Phi_2\,(a_1, a_2, a_3;\, c;\, x, y, z)$. The function ${}_3\Phi\,{}^{(4)}_{B}$ occurs as ${}_3\Phi'$ in [8] and ${}_3\Phi\,{}^{(3)}_{A}$ occurs as Ψ_2 in [4 ; p. 445].

3. VARIOUS EXPANSIONS OF THE FUNCTIONS AND RELATIONS BETWEEN THEM

The thirty-eight confluent functions which we have defined by triple power series may also be represented by power series in x, y or z, by performing the process of confluence on the similar expansions given by Lauricella and Saran for the fourteen F-functions.

We thus find:

(3·1)
$$
\begin{aligned}
&{}_3\Phi_D^{(1)}(a, b_1, b_2; c; x, y, z)\\
&= \sum_{m=0}^{\infty} \frac{(a, m)\,(b_1, m)}{(1, m)\,(c, m)}\, x^m\, \Phi_1(a+m, b_2; c+m; y, z)\\
&= \sum_{n=0}^{\infty} \frac{(a, n)\,(b_2, n)}{(1, n)\,(c, n)}\, y^n\, \Phi_1(a+n, b_1; c+n; x, z)\\
&= \sum_{p=0}^{\infty} \frac{(a, p)}{(1, p)\,(c, p)}\, z^p\, F_1(a+p, b_2, b_3; c+p; x, y).
\end{aligned}
$$

Similar formulae can be written for the other confluent functions.

We shall next consider formulae derived from the definite integral expressions for the F-functions, such as [1, p. 115]:

(3·2)
$$
\begin{aligned}
&F_A(a, b_1, b_2, b_3; c_1, c_2, c_3; x, y, z)\\
&= \frac{\Gamma(c_1)\,\Gamma(c_2)\,\Gamma(c_3)}{\Gamma(b_1)\,\Gamma(b_2)\,\Gamma(b_3)\,\Gamma(c_1-b_1)} \cdot \frac{1}{\Gamma(c_2-b_2)\,\Gamma(c_3-b_3)} \times\\
&\times \int_0^1\int_0^1\int_0^1 u^{b_1-1}\, v^{b_2-1}\, w^{b_3-1}\,(1-u)^{c_1-b_1-1} \times\\
&\times (1-v)^{c_2-b_2-1}\,(1-w)^{c_3-b_3-1}\,(1-ux-vy-wz)^{-a}\, du\, dv\, dw.
\end{aligned}
$$

We have

$$
\begin{aligned}
(1-ux-vy-wz)^{-a} &= (-2)^{-a}\left[1 - \frac{(1-ux)}{2} - \frac{(1-vy)}{2} - \frac{(1-wz)}{2}\right]^{-a}\\
&= (-2)^{-a} \sum_{m,n,p=0}^{\infty} \frac{(a, m+n+p)}{(1, m)\,(1, n)\,(1, p)} \left(\frac{1-ux}{2}\right)^m \left(\frac{1-vy}{2}\right)^n \left(\frac{1-wz}{2}\right)^p.
\end{aligned}
$$

So that

(3·3)
$$
\begin{aligned}
&F_A(a, b_1, b_2, b_3; c_1, c_2, c_3; x, y, z)\\
&= \frac{(-2)^{-a}\,\Gamma(c_1)\,\Gamma(c_2)\,\Gamma(c_3)}{\Gamma(b_1)\,\Gamma(b_2)\,\Gamma(b_3)\,\Gamma(c_1-b_1)\,\Gamma(c_2-b_2)\,\Gamma(c_3-b_3)} \times\\
&\times \sum_{m,n,p=0}^{\infty} \frac{(a, m+n+p)}{(1, m)\,(1, n)\,(1, p)}\, \frac{1}{2^{m+n+p}} \int_0^1 u^{b_1-1}\,(1-u)^{c_1-b_1-1} \times\\
&\times (1-ux)^m\, du \int_0^1 v^{b_2-1}\,(1-v)^{c_2-b_2-1}\,(1-vy)^n\, dv \times\\
&\times \int_0^1 w^{b_3-1}\,(1-w)^{c_3-b_3-1}\,(1-wz)^p\, dz
\end{aligned}
$$

$$= (-2)^{-a} \sum_{m,n,p=0}^{\infty} \frac{(a, m+n+p)}{(1, m)(1, n)(1, p)} \frac{1}{2^{m+n+p}} {}_2F_1(-m, b_1; c_1; x) \times$$

$$\times {}_2F_1(-n, b_2; c_2; y) {}_2F_1(-p, b_3; c_3; z),$$

a formula to which we can apply the process of confluence to obtain the following interesting expansions for the functions ${}_3\Phi^{(1)}A$, ${}_3\Phi^{(2)}A$, and ${}_3\Phi^{(3)}A$:

(3·4) $${}_3\Phi^{(1)}A(a, b_1, b_2; c_1, c_2, c_3; x, y, z)$$

$$= (-2)^{-a} \sum_{m,n,p=0}^{\infty} \frac{(a, m+n+p)}{(1, m)(1, n)(1, p)} \frac{1}{2^{m+n+p}} {}_2F_1(-m, b_1; c_1; x) \times$$

$$\times {}_2F_1(-n, b_2; c_2; y) {}_1F_1(-p; c_3; z).$$

(3·5) $${}_3\Phi^{(2)}A(a, b_1; c_1, c_2, c_3; x, y, z)$$

$$= (-2)^{-a} \sum_{m,n,p=0}^{\infty} \frac{(a, m+n+p)}{(1, m)(1, n)(1, p)} \frac{1}{2^{m+n+p}} {}_2F_1(-m, b_1; c_1; x) \times$$

$$\times {}_1F_1(-n; c_2; y) {}_1F_1(-p; c_3; z).$$

(3·6) $${}_3\Phi^{(3)}A(a; c_1, c_2, c_3; x, y z)$$

$$= (-2)^{-a} \sum_{m,n,p=0}^{\infty} \frac{(a, m+n+p)}{(1, m)(1, n)(1, p)} \frac{1}{2^{m+n+p}} \times$$

$$\times {}_1F_1(-m; c_1; x) {}_1F_1(-n; c_2; y) {}_1F_1(-p; c_3, z).$$

Again, since [5, p. 76]

(3·7) $$F_1(a, b_1, b_2; c; x, y)$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{(a, m)(b_1, m)(b_2, m)}{(1, m)(c-b_2, m)(c, m)} x^m F_2(a+m, b_1+m, b_2+m; c-b_2+m, c+m; x, y)$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(a, n)(b_1, n)(b_2, n)}{(1, n)(c-b_1, n)(c, n)} y^n F_2(a+n, b_1+n, b_2+n; c+n, c-b_1+n; x, y),$$

we obtain the formula:

(3·8) $$FG(a_1, a_1, a_1, b_1, b_2, b_3; c_1, c_2, c_2; x, y, z)$$

$$= \sum_{m=0}^{\infty} \frac{(a_1, m)(b_1, m)}{(1, m)(c_1, m)} x^m F_1(a_1+m, b_2, b_3; c_2, y, z)$$

$$= \sum_{m=0}^{\infty} \frac{(a_1, m)(b_1, m)}{(1, m)(c_1, m)} x^m \sum_{r=0}^{\infty} \frac{(-1)^r (a_1+m, r)(b_2, r)(b_3, r)}{(1, r)(c_2-b_2, r)(c_2, r)} z^r \times$$

$$\times F_2(a_1+m+r, b_2+r, b_3+r; c_2+r, c_2-b_2+r; y, z)$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (a_1, r)(b_2, r)(b_3, r)}{(1, r)(c_2, r)(c_2-b_2, r)} z^r FA(a_1+r, b_1, b_2+r, b_3+r; c_1, c_2+r, c_2-b_2+r; x, y, z).$$

From this result we obtain by confluence :

(3·9) $$ {}_3\Phi^{(1)}G(a_1, a_1, a_1, b_1, b_2 ; c_1, c_2, c_2 ; x, y, z) $$
$$ = \sum_{r=0}^{\infty} \frac{(-1)^r (a_1, r)(b_2, r)}{(1, r)(c_2, r)(c_2 - b_2, r)} z^r \times $$
$$ {}_3\Phi^{(1)}A(a_1 + r, b_1, b_2 + r ; c_1, c_2 + r, c_2 - b_2 + r ; x, y, z) $$

(3·10) $$ {}_3\Phi^{(2)}F(a_1, a_1, a_1, b_2 ; c_1, c_2, c_2 ; x, y, z) $$
$$ = \sum_{r=0}^{\infty} \frac{(-1)^r (a_1, r)(b_2, r)}{(1, r)(c_2, r)(c_2 - b_2, r)} z^r \times $$
$$ {}_3\Phi^{(2)}A(a_1 + r, b_2 + r ; c_1, c_2 + r, c_2 - b_2 + r ; x, y, z) $$

(3·11) $$ {}_3\Phi^{(5)}F(b_2 ; c_1, c_2, c_2 ; x, y, z) $$
$$ = \sum_{r=0}^{\infty} \frac{(-1)^r (b_2, r) z^r}{(1, r)(c_2, r)(c_2 - b_2, r)} \, {}_1F_1(b_2 + r ; c_1 ; x) \times $$
$$ \times \, {}_0F_1(c_2 + r ; y) \, {}_0F_1(c_2 - b_2 + r ; z). $$

Next, using (3·7), we have

(3·12) $$ FF(a_1, a_1, a_1, b_1, b_2, b_1 ; c_1, c_2, c_2 ; x, y, z) $$
$$ = \sum_{m=0}^{\infty} \frac{(a_1, m)(b_1, m)}{(1, m)(c_1, m)} x^m \, F_1(a_1 + m, b_2, b_1 + m ; c_2 ; y, z) $$
$$ = \sum_{m=0}^{\infty} \frac{(a_1, m)(b_1, m)}{(1, m)(c_1, m)} x^m \sum_{r=0}^{\infty} \frac{(-1)^r (a_1 + m, r)(b_2, r)(b_1 + m, r)}{(1, r)(c_2, r)(c_2 - b_2, r)} \times $$
$$ \times \, F_2(a_1 + m + r, b_2 + r, b_1 + m + r ; c_2 + r, c_2 - b_2 + r ; y, z) $$
$$ = \sum_{r=0}^{\infty} \frac{(-1)^r (a_1, r)(b_1, r)(b_2, r)}{(1, r)(c_2, r)(c_2 - b_2, r)} z^r \times $$
$$ \times \, FE(a_1 + r, a_1 + r, a_1 + r, b_1 + r, b_2 + r, b_1 + r ; c_1, c_2 + r, c_2 - b_2 + r ; x, y, z). $$

From this we obtain by confluence :

(3·13) $$ {}_3\Phi^{(4)}F(b_1, b_2, b_1 ; c_1, c_2, c_2 ; x, y, z) $$
$$ = \sum_{r=0}^{\infty} \frac{(-1)^r (b_1, r)(b_2, r)}{(1, r)(c_2, r)(c_2 - b_2, r)} z^r \, \Psi_2(b_1 + r ; c_1, c_2 - b_2 + r ; x, z) $$
$$ \times \, {}_1F_1(b_2 + r ; c_2 + r ; y). $$

The formula (3·10) can also be obtained from (3·12).

Again it can be easily shown that

(3·14) $$ F_3(a_1, a_2, b_1, b_2 ; c ; x, y) $$
$$ = \sum_{m=0}^{\infty} \frac{(-1)^m (a_1, m)(a_2, m)(b_1, m)}{(1, m)(c, m)(c - a_2, m)} x^m \, {}_2F_1(a_1 + m, b_1 + m ; c - a_2 + m ; x) $$
$$ \times \, {}_2F_1(a_2 + m, b_2 ; c + m ; y). $$

Using (3·14), we get

$$(3\cdot15)\qquad F_N(a_1, a_2, a_3 ; b_1, b_2, b_1 ; c_1, c_2, c_2 ; x, y, z)$$
$$= \sum_{m=0}^{\infty} \frac{(a_1, m)\,(b_1, m)}{(1, m)\,(c_1, m)} x^m F_3(a_2, a_3, b_2, b_1 + m ; c_2 ; y, z)$$
$$= \sum_{m=0}^{\infty} \frac{(a_1, m)\,(b_1, m)}{(1, m)\,(c_1, m)} x^m \sum_{r=0}^{\infty} \frac{(-1)^r (a_2, r)\,(a_3, r)\,(b_2, r)}{(1, r)\,(c_2, r)\,(c_2 - a_3, r)} y^r \times$$
$$\times\ {}_2F_1(a_2 + r, b_2 + r ; c_2 - a_3 + r ; y)\ {}_2F_1(a_3 + r, b_1 + m ; c_2 + r ; z)$$
$$= \sum_{r=0}^{\infty} \frac{(-1)^r (a_2, r)\,(a_3, r)\,(b_2, r)}{(1, r)\,(c_2, r)\,(c_2 - a_3, r)} y^r F_2(b_1, a_1, a_3 + r ; c_1, c_2 + r ; x, z)$$
$$\times\ {}_2F_1(a_2 + r, b_2 + r ; c_2 - a_3 + r ; y).$$

From this we obtain by confluence

$$(3\cdot16)\qquad {}_3\Phi^{(1)}{}_N(a_1, a_2, a_3, b_1, b_1 ; c_1, c_2, c_2 ; x, y, z)$$
$$= \sum_{r=0}^{\infty} \frac{(-1)^r (a_2, r)\,(a_3, r)}{(1, r)\,(c_2, r)\,(c_2 - a_3, r)} y^r F_2(b_1, a_1, a_3 + r ; c_1, c_2 + r ; x, z)$$
$$\times\ {}_1F_1(a_2 + r ; c_2 - a_3 + r ; y).$$

$$(3\cdot17)\qquad {}_3\Phi^{(2)}{}_N(a_2, a_3, b_1, b_2, b_1 ; c_1, c_2, c_2 ; x, y, z)$$
$$= \sum_{r=0}^{\infty} \frac{(-1)^r (a_2, r)\,(a_3, r)\,(b_2, r)}{(1, r)\,(c_2, r)\,(c_2 - a_3, r)} y^r \times \Psi_1(b_1, a_3 + r ; c_1, c_2 + r ; x, z) \times$$
$$\times\ {}_2F_1(a_2 + r, b_2 + r ; c_2 - a_3 + r ; y).$$

$$(3\cdot18)\qquad {}_3\Phi^{(3)}{}_N(a_3, b_1, b_2, b_1 ; c_1, c_2, c_2 ; x, y, z)$$
$$= \sum_{r=0}^{\infty} \frac{(-1)^r (a_3, r)\,(b_2, r)}{(1, r)\,(c_2, r)\,(c_2 - a_3, r)} y^r \Psi_1(b_1, a_3 + r ; c_1, c_2 + r ; x, z)$$
$$\times\ {}_1F_1(b_2 + r ; c_2 - a_3 + r ; y).$$

Similar relations can be given for other confluent functions.

It can be easily shown that an important relation exists between ${}_3\Phi^{(1)}{}_D$ and ${}_3\Phi^{(1)}{}_S$ and that, in fact, they also reduce to each other. Starting with the known result [2 ; formula (63)].

$$(3\cdot19)\qquad \Phi_1(a, b ; c ; x, y) = e^y\, \Xi_1(a, c - a, b ; c ; x, -y).$$

We obtain the formula :

$$(3\cdot20)\qquad {}_3\Phi^{(1)}{}_D(a, b_1, b_2 ; c ; x, y, z)$$
$$= \sum_{m=0}^{\infty} \frac{(a, m)\,(b_1, m)}{(1, m)\,(c, m)} x^m \Phi_1(a + m, b_2 ; c + m ; y, z)$$

$$= e^{z} \sum_{m=0}^{\infty} \frac{(a, m)\,(b_1, m)}{(1, m)\,(c, m)}\, x^m\, \Xi_1 (a + m, c - a, b_2 ; c + m ; y, - z)$$

$$= e^{z} \sum_{m,n,p=0}^{\infty} \frac{(a, m + n)\,(c - a, p)\,(b_1, m)\,(b_2, n)}{(1, m)\,(1, n)\,(1, p)\,(c, m + n + p)}\, x^m\, y^n\, z^p\, (-1)^p$$

$$= e^{z}\, {}_3\Phi^{(1)}S\,(a, a, c - a, b_1, b_2 ; c ; x, y, - z).$$

When $x = 0$, (3·20) reduces to a form identical to (3·19).

When $x = y = 0$, (3·20) reduces to

$$(3·21) \qquad {}_1F_1\,(a ; c ; z) = e^{z}\, {}_1F_1(c - a ; c ; - z),$$

which is Kummer's first theorem.

4. DIFFERENTIAL EQUATIONS SATISFIED BY THE FUNCTIONS

The thirty-eight confluent functions satisfy partial differential equations of rather simple forms, which it is easy to obtain, by confluence from the systems of equations found by Lauricella[6] and Saran[9] for the F–functions.

Thus we find that the systems for the function ${}_3\Phi^{(1)}A$ is :

$$x(1 - x)\frac{\partial^2 w}{\partial x^2} - x\left(y\frac{\partial^2 w}{\partial y \partial x} + z\frac{\partial^2 w}{\partial z \partial x}\right) + [c_1 - (a + b_1 + 1)\, x]\frac{\partial w}{\partial x} - b_1\left(y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z}\right) - ab_1 w = 0,$$

$$(4·1) \quad y(1 - y)\frac{\partial^2 w}{\partial y^2} - p\left(z\frac{\partial^2 w}{\partial z \partial y} + x\frac{\partial^2 w}{\partial x \partial y}\right) + [c_2 - (a + b_2 + 1)\, y]\frac{\partial w}{\partial y} - b_2\left(z\frac{\partial w}{\partial z} + x\frac{\partial w}{\partial x}\right) - a\, b_2\, w = 0,$$

$$z\frac{\partial^2 w}{\partial z^2} + (c_3 - z)\frac{\partial w}{\partial z} - x\left(\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y}\right) - a\, w = 0,$$

and similarly for other functions.

The system (4·1) is of the type

$$\frac{\partial^2 w}{\partial x^2} = \alpha_1\frac{\partial^2 w}{\partial y \partial z} + \alpha_2\frac{\partial^2 w}{\partial z \partial x} + \alpha_3\frac{\partial^2 w}{\partial x \partial y} + \alpha_4\frac{\partial w}{\partial x} + \alpha_5\frac{\partial w}{\partial y} + \alpha_6\frac{\partial w}{\partial z} + \alpha_7\, w$$

$$\frac{\partial^2 w}{\partial y^2} = \beta_1\frac{\partial^2 w}{\partial y \partial z} + \beta_2\frac{\partial^2 w}{\partial z \partial x} + \beta_3\frac{\partial^2 w}{\partial x \partial y} + \beta_4\frac{\partial w}{\partial x} + \beta_5\frac{\partial w}{\partial y} + \beta_6\frac{\partial w}{\partial z} + \beta_7\, w$$

$$\frac{\partial^2 w}{\partial z^2} = \gamma_1\frac{\partial^2 w}{\partial y \partial z} + \gamma_2\frac{\partial^2 w}{\partial z \partial x} + \gamma_3\frac{\partial^2 w}{\partial x \partial y} + \gamma_4\frac{\partial w}{\partial x} + \gamma_5\frac{\partial w}{\partial y} + \gamma_6\frac{\partial w}{\partial z} + \gamma_7\, w ;$$

of which a general theory has been given by Lauricella[6].

We may observe that the system satisfied by the function

$${}_3\Phi^{(1)}A(a, b_1, b_2 ; c_1, c_2, c_3 ; x, y, z)$$

$$x^{1-c_1}\ {}_3\Phi^{(1)}A(a+1-c_1,\ b_1+1-c_1;\ 2-c_1,\ c_2,\ c_3;\ x, y, z),$$

$$y^{1-c_2}\ {}_3\Phi^{(1)}A(a+1-c_2,\ b_1,\ b_2+1-c_2;\ c_1,\ 2-c_2,\ c_3;\ x, y, z),$$

$$z^{1-c_3}\ {}_3\Phi^{(1)}A(a+1-c_3,\ b_1,\ b_2,\ c_1,\ c_2,\ 2-c_3;\ x, y, z),$$

$$x^{1-c_1}\ y^{1-c_2}\ {}_3\Phi^{(1)}A(a+2-c_1-c_2,\ b_1+1-c_1,\ b_2+1-c_2;\ 2-c_1,\ 2-c_2,\ c_3;\ x, y, z)$$

$$y^{1-c_2}\ z^{1-c_3}\ {}_3\Phi^{(1)}A(a+2-c_2-c_3,\ b_1,\ b_2+1-c_2;\ c_1,\ 2-c_2,\ 2-c_3;\ x, y, z)$$

$$z^{1-c_3}\ x^{1-c_1}\ {}_3\Phi^{(1)}A(a+2-c_3-c_1,\ b_1+1-c_1,\ b_2;\ 2-c_1,\ c_2,\ 2-c_3;\ x, y, z)$$

and

$$x^{1-c_1}\ y^{1-c_2}\ z^{1-c_3}\ {}_3\Phi^{(1)}A(a+3-c_1-c_2-c_3,\ b_1+1-c_1,\ b_2+1-c_2;\ 2-c_1,\ 2-c_2,\ 2-c_3;\ x, y, z).$$

Similar results may be obtained for the functions ${}_3\Phi^{(1)}A$ and ${}_3\Phi^{(3)}A$, so that the general solutions of the three ${}_3\Phi A$ systems can easily be expressed in terms of the ${}_3\Phi A$ functions themselves.

5. SOME SPECIAL PROPERTIES OF THE CONFLUENT FUNCTIONS

We shall now consider a few formulae illustrative of the properties of the confluent hypergeometric functions of three variables.

The function ${}_3\Phi^{(1)}D$ admits recurrence formulae analogous to the formulae for Φ_1 given by Humbert[5]. Thus

(5·1)
$$\frac{b_1x}{c}\ {}_3\Phi^{(1)}D(a+1,\ b_1+1,\ b_2;\ c;\ x, y, z) + \frac{b_2y}{c}\ {}_3\Phi^{(1)}D(a+1,\ b_1,\ b_2+1;\ c;\ x, y, z) + \frac{z}{c}\ {}_3\Phi^{(1)}D(a+1,\ b_1,\ b_2;\ c;\ x, y, z)$$
$$= {}_3\Phi^{(1)}D(a+1,\ b_1,\ b_2;\ c;\ x, y, z) - {}_3\Phi^{(1)}D(a,\ b_1,\ b_2;\ c;\ x, y, z).$$

(5·2)
$${}_3\Phi^{(1)}D(a,\ b_1+1,\ b_2;\ c;\ x, y, z)$$
$$= {}_3\Phi^{(1)}D(a,\ b_1,\ b_2;\ c;\ x, y, z) + \frac{ax}{c}\ {}_3\Phi^{(1)}D(a+1,\ b_1+1,\ b_2;\ c+1;\ x, y, z)$$

(5·3)
$$
\begin{aligned}
&{}_3\Phi^{(1)}{}_D(a, b_1+1, b_2+1; c; x, y, z)\\
&= {}_3\Phi^{(1)}{}_D(a, b_1, b_2; c; x, y, z)\\
&+ \frac{ax}{c}\,{}_3\Phi^{(1)}{}_D(a+1, b_1+1, b_2; c+1; x, y, z)\\
&+ \frac{ay}{c}\,{}_3\Phi^{(1)}{}_D(a+1, b_1, b_2+1; c+1; x, y, z)\\
&+ \frac{a(a+1)}{c(c+1)}\,x\,y\,{}_3\Phi^{(1)}{}_D(a+2, b_1+1, b_2+1; c+2; x, y, z), \text{ etc.}
\end{aligned}
$$

The relations:

(5·4)
$$
\frac{\partial}{\partial x}\,{}_3\Phi^{(1)}{}_D(a, b_1, b_2; c; x, y, z) = \frac{ab_1}{c}\,{}_3\Phi^{(1)}{}_D(a+1, b_1+1, b_2; c+1; x, y, z)
$$

(5·5)
$$
\frac{\partial^2}{\partial x\partial y}\,{}_3\Phi^{(1)}{}_D(a, b_1, b_2; c; x, y, z) = \frac{a(a+1)\,b_1\,b_2}{c(c+1)}\,{}_3\Phi^{(1)}{}_D(a+2, b_1+1, b_2+1; c+2; x, y, z), \text{ etc.,}
$$

show that the derivatives of the function ${}_3\Phi^{(1)}{}_D$ are expressible in terms of the function itself.

Similar formulae can be obtained for other functions.

Again, a simple definite integral representation for ${}_3\Phi^{(1)}{}_D$ is:

(5·6)
$$
{}_3\Phi^{(1)}{}_D(a, b_1, b_2; c; x, y, z) = \frac{\Gamma(c)}{\Gamma(a)\ \Gamma(c-a)} \times \int_0^1 u^{a-1}\,(1-u)^{c-a-1}\,(1-ux)^{-b_1}\,(1-uy)^{-b_2}\,e^{uz}\,du,
$$

where $Re(c) > Re(a) > 0$.

The result (5·6) is analogous to that for Φ_1 given by Humbert[5]. Similarly, multiple integral representations for the other confluent functions can be derived from the corresponding results for the F-functions. The function ${}_3\Phi^{(3)}{}_A(a; c_1, c_2, c_3; x, y, z)$ is particularly interesting as its properties afford a direct generalisation of the one-variable confluent function. This fact can be exhibited more clearly by substituting for ${}_3\Phi^{(3)}{}_A$ the function $M_{k,\lambda,\mu,\nu}(x, y, z)$ defined by Humbert[1] as:

(5·7)
$$
\begin{aligned}
&M_{k,\lambda,\mu,\nu}(x, y, z)\\
&= x^{\lambda+\frac12}\,y^{\mu+\frac12}\,z^{\nu+\frac12}\,e^{-\frac12(x+y+z)}\,{}_3\Phi^{(3)}{}_A(\lambda+\mu+\nu-k+\tfrac32; 2\lambda+1, 2\mu+1, 2\nu+1; x, y, z)\\
&= x^{\lambda+\frac12}\,y^{\mu+\frac12}\,z^{\nu+\frac12}\,e^{-\frac12(x+y+z)} \sum_{m,n,p=0}^{\infty} \frac{(\lambda+\mu+\nu-k+\frac32,\ m+n+p)\ x^m\,y^n\,z^p}{(1, m)\ (1, n)\ (1, p)\ (2\lambda+1, m)\ (2\mu+1, n)\ (2\nu+1, p)}
\end{aligned}
$$

This function reduces to the Whittaker function $M_{k,\lambda}(x)$ on putting $\mu = -\frac{1}{2}$, $\nu = -\frac{1}{2}$, and $y = 0$, $z = 0$ provided that $y/(2\mu + 1) \to 0$, $z/(2\nu + 1) \to 0$.

Many recurrence formulae can be written for the M-function. We shall only give the following as an example :

$$(5\cdot 8) \qquad x \frac{\partial}{\partial x} M_{k,\lambda,\mu,\nu} = (\lambda + \tfrac{1}{2} - \tfrac{1}{2}x) M_{k,\lambda,\mu,\nu} + x^{\frac{1}{2}} \frac{\lambda+\mu+\nu-k+\frac{3}{2}}{2\lambda+1} M_{k-\frac{1}{2},\, \lambda+\frac{1}{2},\, \mu,\nu}\,.$$

We also get results analogous to the expansion for ${}_3\Phi^{(3)}A$.

Thus

$$(5\cdot 9) \quad M_{k,\lambda,\mu,\nu}(x, y, z) = e^{-x/2} \sum_{m=0}^{\infty} x^{m+\lambda+\frac{1}{2}} \frac{(\lambda+\mu+\nu-k+\frac{3}{2}, m)}{(1, m)\,(2\lambda+1, m)} M_{k-\lambda-m-\frac{1}{2},\, \mu,\nu}(y, z)$$

$$= e^{-y/2} \sum_{m=0}^{\infty} x^{m+\mu+\frac{1}{2}} \frac{(\lambda+\mu+\nu-k+\frac{3}{2}, m)}{(1, m)\,(2\mu+1, m)} M_{k-\mu-m-\frac{1}{2},\, \lambda,\nu}(x, z)$$

$$= e^{-z/2} \sum_{m=0}^{\infty} z^{m+\nu+\frac{1}{2}} \frac{(\lambda+\mu+\nu-k+\frac{3}{2}, m)}{(1, m)\,(2\nu+1, m)} M_{k-\nu-m-\frac{1}{2},\, \lambda,\mu}(x, y).$$

By transforming the formula (3·6), we get

$$(5\cdot 10) \qquad M_{k,\lambda,\mu,\nu}(x, y, z) = (-2)^{k-\lambda-\mu-\nu-\frac{3}{2}} \sum_{m,n,p=0}^{\infty} \left[\frac{(\lambda+\mu+\nu-k+\frac{3}{2}, m+n+p)}{(1, m)\,(1, n)\,(1, p)\, 2^{m+n+p}} \times \right.$$

$$\left. \times\, M_{m+\lambda+\frac{1}{2},\lambda}(x)\, M_{n+\mu+\frac{1}{2},\mu}(y)\, M_{p+\nu+\frac{1}{2},\nu}(z). \right.$$

Now it is known that[5].

$$(5\cdot 11) \qquad M_{m+\lambda+\frac{1}{2},\lambda}(x) = (-1)^m (1, m)\, \Gamma(2\lambda+1)\, x^{\lambda+\frac{1}{2}}\, e^{-x/2}\, T^m_{2\lambda}(x), \text{ etc.}$$

where $T^m_{2\lambda}(x)$ is a polynomial of degree m considered by Sonine[5]. Hence (5·10) can be written as

$$(5\cdot 12) \qquad M_{k,\lambda,\mu,\nu}(x, y, z)$$

$$= (-2)^{k-\lambda-\mu-\nu-\frac{3}{2}}\, x^{\lambda+\frac{1}{2}}\, y^{\mu+\frac{1}{2}}\, z^{\nu+\frac{1}{2}}\, e^{-\frac{1}{2}(x+y+z)}\, \Gamma(2\lambda+1)\, \Gamma(2\mu+1)\, \Gamma(2\nu+1) \times$$

$$\times \sum_{m,n,p=0}^{\infty} (-\tfrac{1}{2})^{m+n+p} (\lambda+\mu+\nu-k+\tfrac{3}{2}, m+n+p)\, T^m_{2\lambda}(x)\, T^n_{2\mu}(y)\, T^p_{2\nu}(z).$$

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ON CERTAIN THEOREMS RELATING TO THE GENERALISED HANKEL TRANSFORM

By

PRATAP SINGH

Department of Mathematics, Faculty of Science, B. I. T. S. (Pilani)

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ABSTRACT

In this paper the author has proved several theorems on self-reciprocal functions, the kernels being $\varpi_{\mu_1, \mu_2, \ldots, \mu_n}(x)$ and $\varpi^{\nu_1, \nu_2, \ldots, \nu_m}_{\mu_1, \mu_2, \ldots, \mu_n}(x)$, $(m < n)$.

A typical theorem is :

Let $f(t) \doteqdot \psi(p)$, $g(t) \doteqdot \phi(p)$ and $x^{\nu-\frac{1}{2}} g(1/x)$ be the $\varpi^{\nu_1, \nu_2, \ldots, \nu_m, \nu}_{\mu_1, \mu_2, \ldots, \mu_n}(x)$ tranrform of $x^{\nu-\frac{1}{2}} f(1/x)$. Then $y^{\nu-\frac{1}{2}} \phi(y)$ shall be the $\varpi^{\nu_1, \nu_2, \ldots, \nu_m}_{\mu_1, \mu_2, \ldots \mu_n, \nu+1}(x)$ transform of $x^{\nu-\frac{1}{2}} \psi(x)$; provided $f(t)$ and $g(t)$ are continuous and absolutely integrable in $(0,\infty)$ and conversely.

INTRODUCTION

Let $\psi(p) = p \int_0^\infty e^{-pt} f(t) \; dt.$ (0·1)

$R(p) > 0$. Then we say that $\psi(p)$ is operationally related to $f(t)$ and symbolically we write $\psi(p) \doteqdot f(t)$ or $f(t) \doteqdot \psi(p)$ (0·2)

Goldstein [3] has proved that if $f(t) \doteqdot \psi(p)$ and $g(t) \doteqdot \phi(p)$,

then
$$\int_0^\infty \phi(t) f(t) \frac{dt}{t} = \int_0^\infty \psi(t) \, g(t) \frac{dt}{t} \tag{0·3}$$

provided the necessary changes in the orders of integrations are permissible and the integrals converge.

Also "if $f(t)$ is a continuous function satisfying (0·1), then it is the only continuous function doing so". This theorem is due to Lerch [4].

Watson [8] has defined a fourier kernel

$$\varpi_{\mu,\nu}(x) = \sqrt{x} \int_0^\infty J_\mu(t) \, J_\nu(x/t) \frac{dt}{t}, \; Re(\mu, \nu) \geqslant -\tfrac{1}{2} \tag{0·4}$$

Bhatnagar, [1] has further generalised the above kernel as

$$\varpi_{\mu_1, \ldots \mu_n}(x) = \int_0^\infty \varpi_{\mu_1, \ldots \mu_n}(x/t) \, J_{\mu_n}(t) \, dt, \; Re(\mu_1, \ldots . \mu_n) \geqslant -\tfrac{1}{2} \tag{0·5}$$

Mainra, [5] has defined the kernel

$$\varpi^{\lambda}_{\mu,\nu}(x) = \int_0^\infty \varpi_{\mu,\nu}(xy)\, J_\lambda(y)\, \sqrt{y}\, dy,\ Re(\mu, \nu, \lambda) \geqslant -\tfrac{1}{2} \tag{0·6}$$

and has established that $\varpi^{\lambda}_{\mu,\nu}(x)$ is a fourier kernel.

Mainra has further extended this kernel to $m + n$ parameters and the generalised kernel is defined as

$$\varpi^{\lambda_1, \ldots \lambda_m}_{\mu_1, \ldots \mu_n}(x) = \int_0^\infty \varpi^{\lambda_1, \ldots \lambda_{m-1}}_{\mu_1, \ldots \mu_n}(xy) J_{\lambda_m}(y)\, \sqrt{y}\, dy \tag{0·7}$$

$Re(\mu_1, \ldots \mu_n, \lambda_1, \ldots \lambda_m) > -\frac{1}{2},\ n > m.$

$$\varpi^{\lambda_1, \ldots \lambda_m}_{\mu_1, \mu_2 \ldots \mu_n}(x) = 0\,(x^{\mu_r + \frac{1}{2}}),\ r = 1, 2, \ldots n \text{ for small } x,$$

and
$$= 0\,(x^{-\frac{n-1}{2n}}) \text{ for large } x.$$

If
$$f(x) = \int_0^\infty \varpi^{\lambda_1, \ldots \lambda_m}_{\mu_1, \ldots \mu_n}(xy)\, f(y)\, dy, \tag{0·8}$$

we say that the function $f(x)$ is self-reciprocal in this generalised transform. We shall denote such functions as

$$R^{\lambda_1, \ldots \lambda_m}_{\mu_1 \ldots \mu_n}$$

Mainra proved the theorem : [5]

Let
$$f(x) \doteqdot \psi\,(p) \tag{1·1}$$

Then
$$p^{-\theta} \int_0^\infty t^{\theta - 1} J_\nu\,(t/p)\, \psi\,(t)\, dt \doteqdot x^\theta \int_0^\infty t^{-\theta-1} J_\nu(x/t)\, f(t)\, dt, \tag{1·2}$$

provided that $t^{-\theta} f(t)$ and $t^{\theta - 1}\ \psi(t)$ are continuous and absolutely integrable in $(0,\infty)$, the conditions being needed for changing the order of integrations.

Let
$$f_1(x) = x^\theta \int_0^\infty t^{-\theta-1} J_\nu\,(x/t)\, f(t)\, dt \tag{1·3}$$

and
$$\psi_1(p) = p^{-\theta} \int_0^\infty t^{\theta-1}\, J_\nu\,(t/p)\, \psi(t)\, dt, \tag{1·4}$$

so that
$$f_1(x) \doteqdot \psi_1(p).$$

On putting $\theta = \theta_1$, $\nu = \nu_1$, $f(x) = f_1(x)$ and $\psi(p) = \psi_1(p)$ in (1·2) we have

$$p^{-\theta_1} \int_0^\infty t^{\theta_1 - 1}\ \psi_1\,(t)\, J_{\nu_1}(t/p)\, dt \doteqdot x^{\theta_1} \int_0^\infty J_{\nu_1}(x/t)\, t^{-\theta_1 - 1} f_1(t)\, dt. \tag{1·5}$$

Substituting for $f_1(t)$ and $\psi_1(t)$ from (1·3) and (1·4) in (1·5) we have

$$p^{-\theta_1} \int_0^\infty t^{\theta_1 - 1} J_{\nu_1}(t/p)\, t^{-\theta}\, dt \int_0^\infty u^{\theta - 1}\, \psi\,(u)\, J_\nu(u/t)\, du$$

$$\doteqdot x^{\theta_1} \int_0^\infty J_{\nu_1}(x/t)\, t^{\theta - \theta_1 - 1}\, dt \int_0^\infty f(u)\, J_\nu\,(t/u)\, \frac{du}{u^{\theta+1}}\ ;$$

$$R(\theta_1 + \nu_1 - \theta) > -\tfrac{3}{2},\ R(\nu - \theta_1 + \theta) > -\tfrac{3}{2},\ R(\theta_1 + \nu_1) > -1.$$

Changing the order of integrations which is permissible under the above conditions of $f(t)$ and $\psi(t)$, we have

$$p^{-\theta_1} \int_0^\infty u^{\theta-1} \psi(u)\, du \int_0^\infty J_{\nu_1}(t/p) J_\nu(u/t) \frac{dt}{t^{\theta-\theta_1+1}}$$

$$\doteqdot x^{\theta_1} \int_0^\infty f(u) \frac{du}{u^{\theta+1}} \int_0^\infty J_{\nu_1}(x/t) J_\nu(t/u) \frac{dt}{t^{\theta_1-\theta+1}}.$$

Taking $\theta = \theta_1$ we have

$$p^{-\theta_1} \int_0^\infty u^{\theta_1-1} \psi(u)\, du \int_0^\infty J_{\nu_1}(t) J_\nu\left(\frac{u}{tp}\right) dt \doteqdot x^{\theta_1} \int_0^\infty f(u) \frac{du}{u^{\theta_1+1}} \cdot \int_0^\infty J_{\nu_1}(x/ut) J_\nu(t) \frac{dt}{t}$$

By the definition of the function $\varpi_{\mu,\nu}(x)$* we have

$$p^{\frac{1}{2}-\theta_1} \int_0^\infty u^{\theta_1-\frac{3}{2}} \psi(u)\, \varpi_{\nu,\nu_1}(u/p)\, du \doteqdot x^{\theta_1-\frac{1}{2}} \int_0^\infty f(u)\, u^{-\theta_1-\frac{1}{2}}\, \varpi_{\nu,\nu_1}(x/u)\, du$$

Let
$$\psi_2(p) = p^{\frac{1}{2}-\theta_1} \int_0^\infty u^{\theta_1-\frac{3}{2}} \psi(u)\, \varpi_{\nu,\nu_1}(u/p)\, du \tag{1·6}$$

and
$$f_2(x) = x^{\theta_1-\frac{1}{2}} \int_0^\infty f(u)\, u^{-\theta-\frac{1}{2}}\, \varpi_{\nu,\nu_1}(x/u)\, du, \tag{1·7}$$

so that
$$f_2(x) \doteqdot \psi_2(p).$$

On writing $\psi_2(p)$ for $\psi(p)$, $f_2(t)$ for $f(t)$ and θ_2 for θ_1, ν_2 for ν_1 in (1·5) we have (under similar conditions of validity)

$$p^{-\theta_2} \int_0^\infty t^{\theta_2-1} \psi_2(t) J_{\nu_2}(t/p)\, dt \doteqdot x^{\theta_2} \int_0^\infty f_2(t) J_{\nu_2}(x/t) \frac{dt}{t^{\theta_2+1}} \tag{1·8}$$

Substituting the value of $\psi_2(p)$ and $f_2(t)$ from (1·6) and (1·7) in (1·8) we have

$$p^{-\theta_2} \int_0^\infty J_{\nu_2}(t/p)\, t^{\theta_2-\theta_1-\frac{1}{2}}\, dt \int_0^\infty u^{\theta_1-\frac{3}{2}} \psi(u)\, \varpi_{\nu,\nu_1}(u/t)\, du$$

$$\doteqdot x^{\theta_2} \int_0^\infty J_{\nu_2}(x/t) \frac{dt}{t^{\theta_2-\theta_1+\frac{3}{2}}} \int_0^\infty f(u)\, \varpi_{\nu,\nu_1}(t/u) \frac{du}{u^{\theta_1+\frac{1}{2}}}$$

Changing the order of integrations on both sides we have

$$p^{-\theta_2} \int_0^\infty u^{\theta_1-\frac{3}{2}} \psi(u)\, du \int_0^\infty t^{\theta_2-\theta_1-\frac{1}{2}} J_{\nu_2}(t/p)\, \varpi_{\nu,\nu_1}(u/t)\, dt$$

$$\doteqdot x^{\theta_2} \int_0^\infty f(u) \frac{du}{u^{\theta_1+\frac{1}{2}}} \int \varpi_{\nu,\nu_1}(t/u)\, J_{\nu_2}(x/t)\, t^{\theta_1-\theta_2-\frac{3}{2}}\, dt.$$

---

*In all the cases *viz.* $\varpi^{\lambda_1, \lambda_2, \ldots \lambda_m}_{\mu_1, \mu_2, \ldots \mu_n}(x)$ we take μ_r, λ_s, $(r = 1, \ldots n, S = 1, \ldots m) > -\frac{1}{2}$ and $m < n$.

On taking $\theta_1 = \theta_2$ and using the definitions of $\varpi_{\nu,\nu_1,\nu_2}(x)$ we have,

$$p^{\frac{1}{2}-\theta_2}\int_0^\infty u^{\theta_2-\frac{3}{2}}\,\psi(u)\,\varpi_{\nu,\nu_1,\nu_2}(u/p)\,du \doteqdot x^{\theta_2-\frac{1}{2}}\int_0^\infty f(u)\,\varpi_{\nu,\nu_1,\nu_2}(x/u)\,\frac{du}{u^{\theta_2+\frac{1}{2}}} \quad (1\cdot9)$$

On taking $\theta_2 - \frac{1}{2} = n$ and after slight modification, we have

$$\int_0^\infty (u/p)^n\,\psi(u)\,\varpi_{\mu_1,\,\mu_2,\,\mu_3}(u/p)\,\frac{du}{u} \doteqdot \int_0^\infty f(u)\left(\frac{x}{u}\right)^n \varpi_{\mu_1,\,\mu_2\,\mu_3}(x/u)\,\frac{du}{u}\,.$$

Proceeding on the same lines we have :

Let $\qquad f(x) \doteqdot \psi(p)$; then

$$\int_0^\infty (u/p)^n\,\psi(u)\,\varpi_{\mu_1,\,\mu_2,\,\ldots,\,\mu_n}(u/p\,\frac{du}{u} \doteqdot \int_0^\infty f(u)\,(x/u)^n\,\varpi_{\mu_1,\,\mu_2,\ldots,\,\mu_n}(x/u)\,\frac{du}{u} \quad (1\cdot10)$$

Again let us denote it as $\qquad f_1(x) \doteqdot \psi_1(p) \qquad (1\cdot11)$

Mainra proved that if $f_1(x) \doteqdot \psi_1(p)$, [5]

Then $$y^{\mu'_1}\int_0^\infty f_1(x)\,J_{\lambda_1}(x/y)\,\frac{dx}{x^{\mu_1'+1}} \doteqdot p^{-\mu'_1}\int_0^\infty \psi_1(x)\,J_{\lambda_1}(p/x)\,x^{\mu'_1-1}\,dx. \quad (1\cdot12)$$

On substituting the values of $f_1(x)$ and $\psi_1(p)$ from (1·11) in (1·12) we have

$$y^{\mu'_1}\int_0^\infty J_{\lambda_1}(x/y)x^{\mu'_1+n-1}dx\int_0^\infty u^{-n-1}f(u)\,\varpi_{\mu_1,\mu_2,\ldots\,\mu_n}(x/u)\,du$$
$$\doteqdot p^{-\mu'_1}\int_0^\infty J_{\lambda_1}(p/x)\,x^{\mu'_1-n-1}\,dx\int_0^\infty u^{n-1}\,\psi(u)\,\varpi_{\mu_1,\mu_2,\ldots\,\mu_n}(u/x)\,du.$$

Changing the order of integrations on both sides, we have

$$y^{\mu'_1}\int_0^\infty u^{-n-1}f(u)\,du\int_0^\infty J_{\lambda_1}(x/y)\,\varpi_{\mu_1\,\ldots\,\mu_n}(x/u)\frac{dx}{x^{\mu'_1-n+1}}$$
$$\doteqdot p^{-\mu'_1}\int_0^\infty u^{n-1}\,\psi(u)\,du\int_0^\infty J_{\lambda_1}(p/x)\,\varpi_{\mu_1,\ldots\,\mu_n}(u/x)\,x^{\mu'_1-n-1}\,dx.$$

The change of the order of integrations are permissible under the above conditions. On putting $n = \mu'_1 + \frac{3}{2}$, we have

$$y^{\mu'_1+\frac{3}{2}}\int_0^\infty u^{-\mu'_1-\frac{5}{2}}f(u)\,\varpi^{\lambda_1}_{\mu_1,\ldots\,\mu_n}(y/u)\,du \doteqdot p^{-\frac{3}{2}-\mu'_1}\int_0^\infty u^{\mu'_1+\frac{1}{2}}$$
$$\psi(u)\,\varpi^{\lambda_1}_{\mu_1,\ldots\,\mu_n}(u/p)\,du.$$

Let us put

$$f_2(y) = y^{\mu'_1+\frac{3}{2}}\int_0^\infty u^{-\mu'_1-\frac{5}{2}}f(u)\,\varpi^{\lambda_1}_{\mu_1,\ldots\,\mu_n}(y/u)\,du, \quad (1\cdot13)$$

$$\psi_2(p) = p^{-\frac{3}{2}-\mu'_1}\int_0^\infty u^{\mu'_1+\frac{1}{2}}\,\psi(u)\,\varpi^{\lambda_1}_{\mu_1,\ldots\,\mu_n}(u/p)\,du. \quad (1\cdot14)$$

On putting $\mu'_1 = \mu'_2$, $\lambda_1 = \lambda_2$, $f_1(y) = f_2(y)$ and $\psi_1(y) = \psi_2(y)$ in (1·12), we have

$$y^{\mu'_2} \int_0^\infty f_2(x) J_{\lambda_2}(x/y) \frac{dx}{x^{\mu'_2+1}} \doteqdot p^{-\mu'_2} \int_0^\infty \psi_2(x) J_{\lambda_2}(p/x) x^{\mu'_2 - 1}\, dx. \qquad (1·15)$$

Substituting the values of $f_2(x)$ and $\psi_2(x)$ from (1·13) and (1·14) in (1·12) we have

$$y^{\mu'_2} \int_0^\infty J_{\lambda_2}(x/y)\, x^{\mu'_1 - \mu'_2 + \frac{1}{2}}\, dx \int_0^\infty u^{-\mu'_1 - \frac{5}{2}} f(u)\, \varpi^{\lambda_1}_{\mu_1, \ldots \mu_n}(x/u)\, du$$

$$\doteqdot p^{-\mu'_2} \int_0^\infty J_{\lambda_2}(p/x)\, x^{\mu'_2 - \mu'_1 - \frac{5}{2}}\, dx \int_0^\infty u^{\mu'_1 + \frac{1}{2}} \psi(u)\, \varpi^{\lambda_1}_{\mu_1, \ldots \mu_n}(u/x)\, du$$

$R(\lambda_2 + \mu'_1 - \mu'_2) > -\frac{3}{2}$, and $R(\mu'_2 - \mu'_1) > 0$.

On changing the order of integrations on both the sides which is permissible if $u^{-\mu'_1 - 2} f(u)$ and $u^{\mu'_1 + \frac{1}{2}} \psi(u)$ are continuous and absolutely integrable in $(0, \infty)$; we have

$$y^{\mu'_2} \int_0^\infty u^{-\mu'_1 - \frac{5}{2}} f(u)\, du \int_0^\infty \varpi^{\lambda_1}_{\mu_1, \ldots \mu_n}(x/u) J_{\lambda_2}(x/y)\, x^{\mu'_1 - \mu'_2 + \frac{1}{2}}\, dx$$

$$\doteqdot p^{-\mu'_2} \int_0^\infty u^{\mu'_1 + \frac{1}{2}} \psi(u)\, du \int_0^\infty \varpi^{\lambda_1}_{\mu_1, \ldots \mu_n}(u/x)\, J_{\lambda_2}(p/x)\, x^{\mu'_2 - \mu'_1 - \frac{5}{2}} dx$$

On taking $\mu'_1 = \mu'_2$, we have

$$y^{\mu'_2 + \frac{3}{2}} \int_0^\infty u^{-\mu'_2 - \frac{5}{2}}\, \varpi^{\lambda_1, \lambda_2}_{\mu_1, \ldots \mu_n}(y/u) f(u)\, du \doteqdot p^{-\frac{3}{2} - \mu'_2} \int_0^\infty u^{\mu'_2 + \frac{1}{2}} \psi(u)\, \varpi^{\lambda_1, \lambda_2}_{\mu_1, \ldots \mu_n}(u/p)\, du,$$

and proceeding on the same lines we can prove that

$$y^{\mu'_n + \frac{3}{2}} \int_0^\infty u^{-\mu'_n - \frac{5}{2}} f(u)\, \varpi^{\lambda_1, \ldots \lambda_m}_{\mu_1, \ldots \mu_n}(y/u)\, du$$

$$\doteqdot p^{-\frac{3}{2} - \mu'_n} \int_0^\infty u^{\mu'_n + \frac{1}{2}} \psi(u)\, \varpi^{\lambda_1, \lambda_2, \ldots \lambda_m}_{\mu_1, \mu_2, \ldots \mu_n}(u/p)\, du.$$

On putting $\mu'_n + \frac{3}{2} = \theta$ we have

$$y^\theta \int_0^\infty u^{-\theta - 1}\, \varpi^{\lambda_1, \ldots \lambda_m}_{\mu_1, \ldots \mu_n}(y/u) f(u)\, du \doteqdot p^{-\theta} \int_0^\infty u^{\theta - 1}\, \varpi^{\lambda_1, \ldots \lambda_m}_{\mu_1, \ldots \mu_n}(u/p)\, \psi(u)\, du,$$

provided $u^{-\theta - \frac{1}{2}} f(u)$ and $u^{\theta - \frac{1}{2}} \psi(u)$ are continuous and absolutely integrable in $(0, \infty)$.

Let $$g(y) = y^\theta \int_0^\infty u^{-\theta - 1}\, \varpi^{\lambda_1, \ldots \lambda_m}_{\mu_1, \ldots \mu_n}(y/u)\, f(u)\, du, \qquad (1·16)$$

and $$\phi(p) = p^{\theta - 1} \int_0^\infty u^{\theta - 1}\, \varpi^{\lambda_1, \ldots \lambda_m}_{\mu_1, \ldots \mu_n}(u/p)\, \psi(u)\, du, \qquad (1·17)$$

so that $g(x) \doteqdot \phi(p)$.

Mitra and Bose[6] proved that if $g(x) \doteqdot \phi(p$ [6].

Then (under certain conditions) *viz*., $y^{-\nu} g(y)$ and $y^{\nu} \phi(y)$ are continuous and absolutely integrable in $(0, \infty)$ and $R(\nu) > -1$,

$$t^{\nu+1} \int_0^\infty J_\nu(ty)\, g(y) \frac{dy}{y^\nu} \doteqdot p^{1-\nu} \int_0^\infty J_{\nu+1}(pz)\, \phi(z)\, z^\nu\, dz.$$

Substituting the values of $g(z)$ and $\phi(z)$ from (1·16) and (1·17), we have

$$t^{\nu+1} \int_0^\infty J_\nu(tz)\, z^{\theta-\nu}\, dz \int_0^\infty u^{-\theta-1} f(u)\, \varpi^{\lambda_1, \ldots \lambda_m}_{\mu_1, \ldots, \mu_n}(z/u)\, du$$

$$\doteqdot p^{1-\nu} \int_0^\infty J_{\nu+1}(pz)\, z^{\nu-\theta}\, dz \int_0^\infty u^{\theta-1} \psi(u)\, \varpi^{\lambda_1, \ldots \lambda_m}_{\mu_1, \ldots \mu_n}(u/z)\, du$$

$R(2\nu - \theta) > -2$, and $R(\theta - \nu) > -\frac{1}{2}$.

On taking $\theta = \nu + \frac{1}{2}$ and changing the order of integrations on both the sides which is permissible under the above conditions ; we have

$$t^{\nu-\frac{1}{2}} \int_0^\infty y^{-\nu-\frac{3}{2}} f(y)\, \varpi^{\lambda_1, \ldots \lambda_m, \nu}_{\mu_1, \ldots \mu_n}\left[\frac{1}{(ty)}\right] dy$$

$$\doteqdot p^{\frac{1}{2}-\nu} \int_0^\infty \varpi^{\lambda_1, \ldots \lambda_m}_{\mu_1, \ldots \mu_n, \nu+1}(py)\, y^{\nu-\frac{1}{2}}\, \psi(y)\, dy, \; (m+1 < n) \; ;$$

or

$$t^{\nu-\frac{1}{2}} \int_0^\infty y^{\nu-\frac{1}{2}} f\left(\frac{1}{y}\right) \varpi^{\lambda_1, \ldots \lambda_m, \nu}_{\mu_1, \ldots \mu_n}(y/t)\, dy$$

$$\doteqdot p^{\frac{1}{2}-\nu} \int_0^\infty y^{\nu-\frac{1}{2}} \psi(y)\, \varpi^{\lambda_1, \ldots \lambda_m}_{\mu_1, \ldots \mu_n, \nu+1}(py)\, dy. \tag{1·18}$$

Let $t^{\nu-\frac{1}{2}} g(1/t)$ be the $\varpi^{\lambda_1, \ldots \lambda_m, \nu}_{\mu_1, \ldots \mu_n}(x)$ transform of $t^{\nu-\frac{1}{2}} f(1/t)$

and let $g(t) \doteqdot \phi(p)$. Then we have

$$\phi(p) = p^{\frac{1}{2}-\nu} \int_0^\infty \varpi^{\lambda_1, \ldots \lambda_m}_{\mu_1, \ldots \mu_n, \nu+1}(px)\, \psi(x)\, x^{\nu-\frac{1}{2}}\, dx \; ;$$

which shows that $p^{\nu-\frac{1}{2}} \phi(p)$ is the $\varpi^{\lambda_1, \ldots \lambda_m}_{\mu_1, \ldots \mu_n, \nu+1}(x)$ transform of $p^{\nu-\frac{1}{2}} \psi(p)$. Thus we state the theorem as

Theorem 1. Let $f(t) \doteqdot \psi(p)$, $g(t) \doteqdot \phi(p)$ and $x^{\nu-\frac{1}{2}} g(1/x)$ be the $\varpi^{\lambda_1, \ldots \lambda_m, \nu}_{\mu_1, \ldots \mu_n}(x)$ transform of $t^{\nu-\frac{1}{2}} f(1/t)$. Then $p^{\nu-\frac{1}{2}} \phi(p)$ will be the $\varpi^{\lambda_1, \ldots \lambda_m}_{\mu_1, \ldots \mu_n, \nu+1}(x)$ transform of $p^{\nu-\frac{1}{2}} \psi(p)$ provided that the integrals for the transforms exist *viz*, if $f(t)$ and $g(t)$ be continuous and absolutely integrable in $(0, \infty)$.

Conversely we state the theorem as follows :

Theorem 2. Let $f(t) \doteqdot \psi(p)$, $g(t) \doteqdot \phi(p)$ and let $p^{\nu-\frac{1}{2}}\,\phi(p)$ be the $\varpi^{\lambda_1, \ldots \lambda_m}_{\mu_1, \ldots \mu_n, \nu+1}$ (x) transform of $p^{\nu-\frac{1}{2}}\,\psi(p)$. Then $t^{\nu-\frac{1}{2}}\,g(1/t)$ will be the $\varpi^{\lambda_1, \ldots \lambda_m, \nu}_{\mu_1, \ldots \mu_n}$ (x), $(m+1<n)$ transform of $t^{\nu-\frac{1}{2}}f(1/t)$ provided $f(t)$ and $g(t)$ be continuous and absolutely integrable in $(0,\infty)$.

2. Further let us suppose that $x^{\nu-\frac{1}{2}}f(1/x)$ is $R^{\lambda_1, \ldots \lambda_m, \nu}_{\mu_1, \ldots \mu_n}$. Then from (1·18) we have

$$f(t) \doteqdot p^{\frac{1}{2}-\nu}\int_0^\infty \varpi^{\lambda_1, \ldots \lambda_m}_{\mu_1 \ldots \mu_n, \nu+1}(py)\,\frac{\psi(y)}{y^{\frac{1}{2}-\nu}}\,dy.$$

But $f(t) \doteqdot \psi(p)$ then we have

$$p^{\nu-\frac{1}{2}}\,\psi(p) = \int_0^\infty \varpi^{\lambda_1, \ldots \lambda_m}_{\mu_1, \ldots \mu_n, \nu+1}(py)\,\psi(y)\,y^{\nu-\frac{1}{2}}\,dy,$$

which shows that $x^{\nu-\frac{1}{2}}\,\psi(x)$ is $R^{\lambda_1, \ldots \lambda_m}_{\mu_1, \ldots \mu_n, \nu+1}$. Hence we state the theorems.

Theorem 3. Let $f(x) \doteqdot \psi(p)$ and let $x^{\nu-\frac{1}{2}}f(1/x)$ be $R^{\lambda_1, \ldots \lambda_m, \nu}_{\mu_1, \ldots \mu_n}$. Then $x^{\nu-\frac{1}{2}}$ $\psi(x)$ will be $R^{\lambda_1, \ldots \lambda_m}_{\mu_1, \ldots \mu_n, \nu+1}$ provided $t^{-\nu-1}f(t)$ and $t^\nu\,\psi(t)$ are continuous and absolutely integrable in $(0, \infty)$.

Theorem 4. Let $f(x) \doteqdot \psi(p)$ and let $x^{\nu-\frac{1}{2}}\,\psi(x)$ be $R^{\lambda_1, \ldots \lambda_m}_{\mu_1, \ldots \mu_n, \nu+1}$ $(m+1<n)$. Then $x^{\nu-\frac{1}{2}}f(1/x)$ will be $R^{\lambda_1, \ldots \lambda_m, \nu}_{\mu_1, \ldots \mu_n}$ provided that $t^{-\nu-1}\,f(t)$ and $t^\nu\,\psi(t)$ are continuous and absolutely integrable in $(0, \infty)$.

3. From (1·18) we have, if $f_1(x) \doteqdot \psi_1(p)$ then

$$t^{\nu-\frac{1}{2}}\int_0^\infty y^{\nu-\frac{1}{2}}f_1(1/y)\,\varpi^{\lambda_1, \ldots \lambda_m, \nu}_{\mu_1, \ldots \mu_n}(y/t)\,dy \doteqdot p^{\frac{1}{2}-\nu}\int_0^\infty \frac{\psi_1(y)}{y^{\frac{1}{2}-\nu}}\,\varpi^{\lambda_1, \ldots \lambda_m}_{\mu_1, \ldots \mu_n, \nu+1}(py)\,dy \quad (3·1)$$

Let $f(x) \doteqdot \psi(p)$; Then we have $y^n f(y) \doteqdot (-1)^n\,p\,\dfrac{d^n}{dp^n}\,[\psi(p)]$. [2]

$$\text{Let } f_1(y) = y^n f(y) \text{ and } \psi_1(p) = (-1)^n\,p\,\frac{d^n}{dp^n}[\psi(p)] \quad (3·2)$$

On substituting the values of $f_1(t)$ and $\psi_1(t)$ from (3·2) in (3·1), we have

$$t^{\nu-\frac{1}{2}}\int_0^\infty y^{\nu-n-\frac{1}{2}}\,f(1/y)\,\varpi^{\lambda_1, \ldots \lambda_m, \nu}_{\mu_1 \ldots \mu_n}(y/t)\,dy$$

$$\doteqdot p^{\frac{1}{2}-\nu}\int_0^\infty y^{\nu+\frac{1}{2}}\,(-1)^n\,\frac{d^n}{dy^n}\,[\psi(y)]\,\varpi^{\lambda_1, \ldots \lambda_m}_{\mu_1, \ldots \mu_n, \nu+1}(py)\,dy.$$

Let $t^{\nu-\frac{1}{2}}\, g(1/t)$ be the $\varpi^{\lambda_1, \ldots \lambda_m, \nu}_{\mu_1, \ldots \mu_n}(x)$ transform of $y^{\nu-n-\frac{1}{2}}\, f(1/y)$ and let $g(t) \doteqdot \phi(p)$; then we have

$$p^{\nu-\frac{1}{2}}\, \phi(p) = \int_0^\infty (-1)^n\, y^{\nu+\frac{1}{2}}\, \frac{d^n}{dy^n}\,[\psi(y)]\, \varpi^{\lambda_1 \ldots \lambda_m}_{\mu_1 \ldots \mu_n, \nu+1}(py)\, dy$$

which shows that $x^{\nu-\frac{1}{2}}\, \phi(x)$ will be the $\varpi^{\lambda_1, \ldots \lambda_m}_{\mu_1, \ldots \mu_n, \nu+1}(x)$ transform of $y^{\nu+\frac{1}{2}}\,(-1)^n \frac{d^n}{dy^n}[\psi(y)]$ Hence we state the theorem as

Theorem 5. Let $f(x) \doteqdot \psi(p)$ and $g(x) \doteqdot \psi(p)$ and let $x^{\nu-\frac{1}{2}}\, g(1/x)$ be the $\varpi^{\lambda_1, \ldots \lambda_m, \nu}_{\mu_1, \ldots \mu_n}(x)$ transform of $y^{\nu-n-\frac{1}{2}}\, f(1/y)$. Then $x^{\nu-\frac{1}{2}}\, \phi\,(x)$ will be the $\varpi^{\lambda_1, \ldots \lambda_m}_{\mu_1, \ldots \mu_n, \nu+1}(x)$ transform of $(-1)^n\, y^{\nu+\frac{1}{2}} \frac{d^n}{dy^n}[\psi(y)]$ provided that $t^{n-\nu-1}\, f(t)$ and $y^{\nu+\frac{1}{2}} \frac{d^n}{dy^n}[\psi(y)]$ are continuous and absolutely integrable in $(0, \infty)$.

4. From (1·18), we have if $f_1(x) \doteqdot \psi_1(p)$, then

$$t^{\nu-\frac{1}{2}} \int_0^\infty x^{\nu-\frac{1}{2}}\, f_1(1/x)\, \varpi^{\lambda_1, \ldots \lambda_m, \nu}_{\mu_1, \ldots \mu_n}(x/t)\, dx \doteqdot$$

$$p^{\frac{1}{2}-\nu} \int_0^\infty \psi_1(y)\, y^{\nu-\frac{1}{2}}\, \varpi^{\lambda_1, \ldots \lambda_m}_{\mu_1, \ldots \mu_n, \nu+1}(py)\, dy \tag{4·1}$$

Let $f(x) \doteqdot \psi(p)$; then $p\,\psi(p) \doteqdot \frac{d}{dt}[f(t)]$ if $f(0) = 0$. [2]

Let $f_1(t) = \frac{d}{dt}[f(t)]$ and $\psi_1(p) = p\,\psi(p)$; then $f_1(1/t) = -t^2 \frac{d}{dt}[f(1/t)]$.

On substituting these values of $f_1(t)$ and $\psi_1(p)$ in (4·1) we have

$$-t^{\nu-\frac{1}{2}} \int_0^\infty x^{\nu+\frac{3}{2}} \frac{d}{dx}[f(1/x)]\, \varpi^{\lambda_1, \ldots \lambda_m, \nu}_{\mu_1, \ldots \mu_n}(x/t)\, dx$$

$$\doteqdot p^{\frac{1}{2}-\nu} \int_0^\infty \varpi^{\lambda_1, \lambda_2, \ldots \lambda_m}_{\mu_1, \ldots \mu_n, \nu+1}(px)\, x^{\nu+\frac{1}{2}}\, \psi(x)\, dx.$$

Let $x^{\nu-\frac{1}{2}}\phi(x)$ be the $\varpi^{\lambda_1,\ldots\lambda_m}_{\mu_1,\ldots\mu_n,\nu+1}(x)$ transform of $x^{\nu+\frac{1}{2}}\psi(x)$ and let $g(t) \doteqdot \phi(p)$; then we have

$$-t^{\frac{1}{2}-\nu}g(t) = \int_0^\infty x^{\nu+\frac{3}{2}}\frac{d}{dx}[f(1/x)]\ \varpi^{\lambda_1,\ldots\lambda_{m,\nu}}_{\mu_1,\ldots\mu_n}(x/t)\,dx$$

which shows that $-x^{\nu-\frac{1}{2}}g(1/x)$ is the $\varpi^{\lambda_1,\ldots\lambda_{m,\nu}}_{\mu_1,\ldots\mu_n}(x)$ transform of $x^{\nu+\frac{3}{2}}\frac{d}{dx}[f(1/x)]$.

Hence we state the theorem as follows :

Theorem 6. Let $f(x) \doteqdot \psi(p)$ and $g(x) \doteqdot \phi(p)$ and let $x^{\nu-\frac{1}{2}}\phi(x)$ be the $\varpi^{\lambda_1,\ldots\lambda m}_{\mu_1,\ldots\mu_n,\nu+1}(x)$ transform of $y^{\nu+\frac{1}{2}}\psi(y)$. Then $-x^{\nu-\frac{1}{2}}g(1/x)$ will be the $\varpi^{\lambda_1,\ldots\lambda m,\nu}_{\mu_1,\ldots\mu_n}(x)$ transform of $x^{\nu+\frac{3}{2}}\frac{d}{dx}[f(1/x)]$, when $f(0)=0$, provided $t^{\nu+\frac{1}{2}}\psi(t)$ and $t^{\nu+\frac{3}{2}}\frac{d}{dt}[f(1/t)]$ are continuous and absolutely integrable in $(0,\infty)$.

5. Again we have from (1·18) :

Let $f(x) \doteqdot \psi(p)$; then

$$t^{\nu-\frac{1}{2}}\int_0^\infty \varpi^{\lambda_1,\ldots\lambda_{m,\nu}}_{\mu_1,\ldots\mu_n}(x/t)\,f(1/x)\,x^{\nu-\frac{1}{2}}\,dx$$

$$\doteqdot p^{1/2-\nu}\int_0^\infty \varpi^{\lambda_1,\ldots\lambda_m}_{\mu_1,\ldots\mu_n,\nu+1}(px)\,x^{\nu-\frac{1}{2}}\psi(x)\,dx \tag{5·1}$$

Let $p^{\nu-\frac{1}{2}}\phi(1/p)$ be the $\varpi^{\lambda_1,\ldots\lambda_m}_{\mu_1,\ldots\mu_n,\nu+1}(x)$ transform of $x^{\nu-\frac{1}{2}}\psi(x)$.

Then we have

$$\phi(1/p) \doteqdot t^{\nu-\frac{1}{2}}\int_0^\infty \varpi^{\lambda_1,\ldots\lambda_{m,\nu}}_{\mu_1,\ldots\mu_n}(x/t)\,f(1/x)\,x^{\nu-\frac{1}{2}}\,dx \tag{5·2}$$

Let $g(t) \doteqdot \phi(p)$ where $g(t)$ is continuous and absolutely integrable in $(0,\infty)$; then

$$\phi(1/p) \doteqdot \sqrt{t}\int_0^\infty \frac{g(x)}{\sqrt{x}}[J_1(2\sqrt{xt})]\,dx \ [2] \tag{5·3}$$

Hence from (5·2) and (5·3) we have

$$t^{\nu-\frac{1}{2}}\int_0^\infty \varpi^{\lambda_1,\ldots\lambda_m,\nu}_{\mu_1,\ldots\mu_n}(x/t)\, x^{\nu-\frac{1}{2}} f(1/x)\, dx = \sqrt{t}\int_0^\infty x^{-\frac{1}{2}} g(x)\, [J_1(2\sqrt{xt})]\, dx$$

provided both the sides are continuous functions of t, which shows that the $\varpi^{\lambda_1,\ldots\lambda_m,\nu}_{\mu_1,\ldots\mu_n}(x)$ transform $(m+1<n)$ of $y^{\nu-\frac{1}{2}} f(1/y)$ will be

$$x^{\nu-1}\int_0^\infty g(t)\, t^{-\frac{1}{2}} J_1(2\sqrt{t/x})\, dx.$$

Hence we state the theorem as :

Theorem 7. Let $f(x) \doteqdot \psi(p)$ and $g(x) \doteqdot \phi(p)$ and let $x^{\nu-\frac{1}{2}}\ \phi(1/x)$ be the $\varpi^{\lambda_1,\ldots\lambda_m}_{\mu_1,\ldots\mu_n,\nu+1}(x)$ transform of $x^{\nu-\frac{1}{2}}\psi(x)$. Then the $\varpi^{\lambda_1,\ldots\lambda_m,\nu}_{\mu_1,\ldots\mu_n}(x)$ transform of $y^{\nu-\frac{1}{2}}\ f(1/y)$ will be $x^{\nu-1}\int_0^\infty g(t)\ \sqrt{\frac{1}{t}}\ J_1(2\sqrt{t/x})\, dx$ provided $t^\nu\psi(t)$ and $t^{-\nu-1}f(t)$ are continuous and absolutely integrable in $(0, \infty)$.

6. Again from (1·18) we have

$$t^{\nu-\frac{1}{2}}\int_0^\infty \varpi^{\lambda_1,\ldots\lambda_m,\nu}_{\mu_1,\ldots\mu_n}(x/t) f(1/x)\frac{dx}{x^{\frac{1}{2}-\nu}} \doteqdot p^{\frac{1}{2}-\nu}\int_0^\infty \varpi^{\lambda_1,\ldots\lambda_m}_{\mu_1,\ldots\mu_n,\nu+1}(px)$$

$$\psi(x)\ x^{\nu-\frac{1}{2}}\, dx. \tag{6·1}$$

Let $x^{\nu-\frac{1}{2}} g(x)$ be the $\varpi^{\lambda_1,\ldots\lambda_m,\nu}_{\mu_1,\ldots\mu_n}(x)$ transform of $x^{\nu-\frac{1}{2}} f(1/x)$.

Then we have

$$g(1/t) \doteqdot p^{\frac{1}{2}-\nu}\int_0^\infty \varpi^{\lambda_1,\ldots\lambda_m}_{\mu_1,\ldots\mu_n,\nu+1}(px)\, x^{\nu-\frac{1}{2}}\psi(x)\, dx \tag{6·2}$$

Let $g(t) \doteqdot \phi(p)$ where $g(t)$ and $g(1/t)$ are continuous and absolutely integrable in $(0,\infty)$.

Then $g(1/t) \doteqdot \sqrt{p}\int_0^\infty x^{-\frac{1}{2}} J_1(2\sqrt{xp})\,\phi(x)\, dx$, [2]; (6·3)

From (6·2) and (6·3) we have

$$\int_0^\infty x^{-\frac{1}{2}}\phi(x) J_1[(2\sqrt{xp})]\, dx = p^{-\nu}\int_0^\infty \varpi^{\lambda_1,\ldots\lambda_m}_{\mu_1\ldots\mu_n,\nu+1}(px)\, x^{\nu-\frac{1}{2}}\psi(x)\, dx:$$

which shows that the $\varpi^{\lambda_1 \ldots \lambda_m}_{\mu_1, \ldots \mu_n, \nu+1}(x)$ transform of $t^{\nu - \frac{1}{2}} \psi(t)$ will be

$$x^{\nu} \int_0^{\infty} \phi(t)\, t^{-\frac{1}{2}} J_1[(2\sqrt{xt})]\, dt.$$

Hence we state the theorem as follows.

Theorem 8. Let $f(x) \doteqdot \psi(p)$ and $g(x) \doteqdot \phi(p)$ and let $x^{\nu-\frac{1}{2}} g(x)$ be the $\varpi^{\lambda_1, \ldots \lambda_m, \nu}_{\mu_1, \ldots \mu_n}(x)$ transform of $x^{\nu-\frac{1}{2}} f(1/x)$. Then the $\varpi^{\lambda_1, \ldots \lambda_m}_{\mu_1, \ldots \mu_n, \nu+1}(x)$ transform of $x^{\nu-\frac{1}{2}} \psi(x)$ will be

$$x^{\nu} \int_0^{\infty} t^{-\frac{1}{2}} \phi(t) J_1(2\sqrt{xt})\, dt$$

provided $t^{-\nu-1} f(t)$ and $t^{\nu} \psi(t)$ are continuous and absolutely integrable in $(0, \infty)$.

Examples.

1. Let $x^{\nu - \frac{1}{2}} \psi(x) = \sqrt{x}\, K_{\mu+\frac{1}{2}}(\sqrt{x})\, K_{\mu-\frac{1}{2}}(\sqrt{x})$

which is $\qquad R^{0,1}_{-\frac{1}{2}, \mu, \mu+1, -\mu, 1-\mu; \frac{1}{2},}$ [7].

On taking $\nu = -\frac{1}{2}$, we have

$$\psi(p) = p^{\frac{3}{2}}\; K_{\mu+\frac{1}{2}}(\sqrt{p})\; K_{\mu-\frac{1}{2}}(\sqrt{p}).$$

$$\text{Then } f(t) = \frac{\sqrt{\pi}}{2\sqrt{2}} t^{-1} e^{-\frac{1}{2}t^{-1}}\; W_{\frac{1}{2}, \mu}\left(\frac{1}{t}\right)$$

$$\text{Therefore } t^{\nu - \frac{1}{2}} f(1/t) = \frac{\sqrt{\pi}}{2\sqrt{2}} t^{\nu+\frac{1}{2}}\, e^{-\frac{t}{2}}\; W_{\frac{1}{2}, \mu}(t),$$

$$\text{or,} \qquad \frac{1}{t} f(1/t) = \frac{\sqrt{\pi}}{2\sqrt{2}}\, e^{-\frac{1}{2}t}\; W_{\frac{1}{2}, \mu}(t)$$

Hence from theorem 4, we have $e^{-\frac{1}{2}t}\, W_{\frac{1}{2},\mu}(t)$ is $R^{0,1}_{\mu, \mu+1, -\mu, 1-\mu}$.

$$-\tfrac{1}{2} \leqslant \mu \leqslant \tfrac{1}{2}$$

2. Let $x^{\nu - \frac{1}{2}} \psi(x) = x^{2\nu_1} [K_{2\nu_1}(x)]^2$ which is

$R^{2\nu_1, 2\nu_1+1}_{-\frac{1}{2}, \frac{1}{2}, 4\nu_1 - \frac{1}{2}, 4\nu_1+\frac{1}{2}, 2\nu_1 - \frac{1}{2}, 2\nu_1+\frac{1}{2}}$ [7].

Taking $\nu = -\frac{1}{2}$ we have, $\psi(p) = p^{2\nu_1+1}\ [K_{2\nu_1}(p))]^2$

$$\text{Then } f(t) = \frac{\sqrt{\pi}}{2}\ t^{-3\nu_1-\frac{1}{2}}\ e^{-\frac{1}{2t}}\ W_{\nu_1,\nu_1}(1/t)$$

or

$$\frac{1}{t}\ f(1/t) = t^{3\nu_1-\frac{1}{2}}\ e^{-\frac{1}{2}t}\ W_{\nu_1,\nu_1}(t)$$

Hence from theorem 4, we have $t^{3\nu_1-\frac{1}{2}}\ e^{-\frac{1}{2}t}\ W_{\nu_1,\nu_1}(t)$ is $R^{\,2\nu_1,2\nu_1+1}_{\,4\nu_1-\frac{1}{2},4\nu_1+\frac{1}{2},2\nu_1+\frac{1}{2},2\nu_1-\frac{1}{2}}$

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SOME PROPERTIES OF $\varpi_{\mu,\nu}(x)$ AND ITS APPLICATIONS

By

G. K. GOYAL

Department of Mathematics, University of Rajasthan, Jaipur

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ABSTRACT

In this note certain properties of the Fourier kernal $\varpi_{\mu,\nu}(x)$ are given and some integrals involving $\varpi_{\mu,\nu}(x)$ have been evaluated which are used in the next section for obtaining the sums of a few infinite series.

Recently the function $\varpi_{\mu,\nu}(x)$ has been studied by K. P. Bhatnagar, V. P. Mainra and B. Singh regarding its properties and evaluation of some infinite series. This paper deals with a general study of some more properties of $\varpi_{\mu,\nu}(x)$ (given in section 2) from which many results given by B. Singh can be derived as particular cases. In section 3 a list of integrals involving $\varpi_{\mu,\nu}(x)$ is given which are used in section 4 in deducing a few more expansions for G-function and Bessel function $J_\nu(x)$.

1. The Fourier kernal $\varpi_{\mu,\nu}(x)$ which was suggested by Watson in 1931, has been defined by the integral equation

$$(1\cdot1) \qquad \varpi_{\mu,\nu}(x) = \sqrt{x} \int_0^\infty J_\nu(x/t) J_\mu(t) \frac{dt}{t}$$

2. The following properties of $\varpi_{\mu,\nu}(x)$ will be established here

$$(2\cdot1) \qquad \mu\, \varpi_{\mu,\nu}(x) - (\mu+2)\, \varpi_{\mu+2,\nu}(x) + (\mu+4)\, \varpi_{\mu+4,\nu}(x) - \cdots\cdots$$
$$= \frac{\sqrt{x}}{4} G^{2\,0}_{0\,4}\left(\frac{x^2}{16}\,\middle|\, \frac{\nu}{2}, \frac{\mu}{2}, 1-\frac{\mu}{2}, -\frac{\nu}{2}\right)$$

where $x > 0$, $R(\nu + \frac{1}{2}) > 0$, $R(\mu + \frac{3}{2}) > 0$.

$$(2\cdot2) \qquad \frac{\mu}{2}\, \varpi_{\mu,\nu}(x) - (\mu + 2)\, \varpi_{\mu+2,\nu}(x) + (\mu + 4)\, \varpi_{\mu+4,\nu}(x) - \cdots\cdots$$
$$= \frac{\sqrt{x}}{4}\left\{ G^{2\,0}_{0\,4}\left(\frac{x^2}{16}, \middle| \frac{\nu}{2}, \frac{\mu}{2}, 1-\frac{\mu}{2}, -\frac{\nu}{2}\right) - G^{2\,0}_{0\,4}\left(\frac{x^2}{16}\,\middle|\, \frac{\nu}{2}, 1+\frac{\mu}{2}, -\frac{\mu}{2}, -\frac{\nu}{2}\right)\right\}$$

where $x > 0$, $R(\nu + \frac{1}{2}) > 0$, $R(\mu + \frac{3}{2}) > 0$.

$$(2\cdot3) \qquad \varpi_{0,\nu}(x) + 2\cos 2\phi\, \varpi_{2,\nu}(x) + 2\cos 4\phi\, \varpi_{4,\nu}(x) + - \cdots\cdots$$
$$= \sqrt{x}\left\{ \frac{\pi^{\frac{1}{2}} \sin^\nu \phi\, \Gamma(-\frac{\nu}{2})}{4^{\nu+\frac{1}{2}}\, \Gamma(\nu+1)\, \Gamma(\frac{1+\nu}{2})}\, {}_0F_3\left(\nu+1, 1+\frac{\nu}{2}, \frac{1+\nu}{2}, \frac{x^2 \sin^2\phi}{16}\right) \right.$$
$$\left. + \frac{1}{\nu}\, {}_0F_3\left(\tfrac{1}{2}, 1-\frac{\nu}{2}, 1+\frac{\nu}{2}; \frac{x^2\sin^2\phi}{16}\right)\right\}$$

$$(2{\cdot}4) \qquad \varpi_{0,\nu}(x) - 2\cos 2\phi\, \varpi_{2,\nu}(x) + 2\cos 4\phi\, \varpi_{4,\nu}(x) - \ldots\ldots$$

$$= \sqrt{x}\left\{\frac{\pi^{\frac{1}{2}}\cos^\nu\phi\,\Gamma(-\frac{\nu}{2})}{4^{\nu+\frac{1}{2}}\Gamma(1+\nu)\,\Gamma(\frac{1+\nu}{2})}\ {}_0F_3\left(\nu+1, 1+\frac{\nu}{2}, \frac{1+\nu}{2}; \frac{x^2\cos^2\phi}{16}\right)\right.$$

$$\left.+\frac{1}{\nu}\ {}_0F_3\left(\tfrac{1}{2}, 1-\frac{\nu}{2}, 1+\frac{\nu}{2}; \frac{x^2\cos^2\phi}{16}\right)\right\}$$

$$(2{\cdot}5) \qquad 2\sin\phi\,\varpi_{1,\nu}(x) + 2\sin 3\phi\,\varpi_{3,\nu}(x) + 2\sin 5\phi\,\varpi_{5,\nu}(x) + -\ldots\ldots$$

$$= \sqrt{x}\left\{\frac{\pi^{\frac{1}{2}}(x\sin\phi)^\nu\,\Gamma(\frac{1-\nu}{2})}{4^{\nu+\frac{1}{2}}\Gamma(1+\frac{\nu}{2})\,\Gamma(1+\nu)}\ {}_0F_3\left(1+\nu, \frac{1+\nu}{2}, 1+\frac{\nu}{2}; \frac{x^2\sin^2\phi}{16}\right) + \frac{x\sin\phi}{\nu^2-1}\right.$$

$$\left.{}_0F_3\left(\frac{3}{2}, \frac{3-\nu}{2}, \frac{3+\nu}{2}; \frac{x^2\sin^2\phi}{16}\right)\right\}$$

$$(2{\cdot}6) \qquad 2\cos\phi\,\varpi_{1,\nu}(x) - 2\cos 3\phi\,\varpi_{3,\nu}(x) + 2\cos 5\phi\,\varpi_{5,\nu}(x) - \ldots\ldots$$

$$= \sqrt{x}\left\{\frac{\pi^{\frac{1}{2}}(x\cos.\phi)^\nu\,\Gamma(\frac{1-\nu}{2})}{4^{\nu+\frac{1}{2}}\Gamma(\frac{1+\nu}{2})\,\Gamma(1+\nu)}\ {}_0F_3\left(\nu+1, \frac{1+\nu}{2}, 1+\frac{\nu}{2}; \frac{x^2\cos^2\phi}{16}\right) + \frac{x\cos\phi}{\nu^2-1}\right.$$

$$\left.{}_0F_3\left(\frac{3}{2}, \frac{3-\nu}{2}, \frac{3+\nu}{2}; \frac{x^2\cos^2\phi}{16}\right)\right\}$$

$$(2{\cdot}7) \qquad \varpi_{\mu,\nu}(x) + (\mu+2)\,\varpi_{\mu+2,\nu}(x) + \frac{(\mu+4)\,(\mu+1)}{\lfloor 2}\,\varpi_{\mu+4,\nu} + -\ldots\ldots$$

$$= \left(\frac{x}{4}\right)^{\mu+\frac{1}{2}} \frac{\Gamma(\frac{\mu-\nu}{2})}{\Gamma(\mu+1)\,\Gamma(1+\frac{\mu+\nu}{2})}$$

where $x > 0$, $R(\mu+\frac{3}{2}) > 0$, $R(\mu-\nu) < 0$.

Proof : Multiply both the sides of

$$\tfrac{t}{2}J_\mu(t) = \mu J_\mu(t) - (\mu+2)\,J_{\mu+2}(t) + (\mu+4)\,J_{\mu+4}(t) - \ldots\ldots.$$

(4, p. 18) by $\sqrt{x}\,J_\nu(x/t)\,\frac{1}{t}$ and then integrate term by term in the interval $(0,\infty)$ with the help of (1·1) to arrive at the required result (2·1). Similarly by the application of various results given in (4, pp. 18, 32, 33) we obtain the remaining results.

The operation of successive integration term by term is justified as the series is uniformly convergent and each term is integrable.

Here if we put $\mu=1$ in (2·1), $\phi=0^0$ in (2·3) and (2·4) as well as $\mu=0$ and $\mu=2$ in (2·7) we obtain results given by B. Singh.

3. We give below a list of integrals derivable from (1·1) and integrals given in 'The Tables of Integral Transforms', which will be used in the next section for summing up certain infinite series.

$$(3{\cdot}1) \qquad \int_0^\infty x^{-\frac{1}{2}}(x^2+a^2)^{-\frac{1}{2}}\,\varpi_{\mu,\nu}(x)\,dx = \frac{1}{4\sqrt{\pi}}\,G^{3\,1}_{1\,5}\left(\frac{a^2}{16}\,\middle|\,\begin{matrix}\frac{1}{2}\\ 0, \frac{\nu}{2}, \frac{\mu}{2}, -\frac{\mu}{2}, -\frac{\nu}{2}\end{matrix}\right)$$

where $R(\nu+1) > 0$, $R(\mu+1) > 0$, $|\arg a^2| < \pi$.

$$(3{\cdot}2) \qquad \int_0^\infty x^{2\rho-\frac{1}{2}}\, G^{m\,n}_{p\,q}\left(\lambda x^2 \Big|{\alpha_i \atop \beta_j}\right) \varpi_{\mu,\nu}(x)\,dx$$

$$=2^{4\rho}\, G^{m,\,n+2}_{p+4,\,q}\left(16\lambda \Big| {\frac{1-\mu}{2}-\rho\,,\ \frac{1-\nu}{2}-\rho\,,\ \alpha_i,\ \frac{1+\nu}{2}-\rho\,,\ \frac{1+\mu}{2}-\rho \atop \beta_j}\right)$$

where $p+q<2(m+n)$, $-\frac{1}{2}<R(\nu)<\frac{1}{2}$, $|\arg\lambda|<\left(m+n-\frac{p+q}{2}\right)\frac{\pi}{4}$,

$R(\alpha_k+\rho)<\frac{3}{4}\ (k=1,2,\ldots,n+1)$, $R(\beta_s+\rho+\frac{\mu}{2})<-\frac{1}{2}\ (s=1,2\ldots,m)$.

$$(3{\cdot}3) \qquad \int_0^\infty x^{\frac{\nu-\mu+1}{2}}\, J_{\frac{\mu+\nu}{2}}(bx)\, \varpi_{\mu,\nu}(x)\,dx = b^{\mu-\nu-2}\, J_{\frac{\mu+\nu}{2}}\left(\frac{1}{b}\right)$$

where $R(\mu)>R(\nu)>-\frac{3}{2}$, $b>0$.

$$(3{\cdot}4) \qquad \int_0^\infty x^{-\frac{1}{2}}\, H_{-\frac{1}{2}}(ax)\, \varpi_{\frac{1}{2},\nu}(x)\,dx = \frac{1}{a^2} J_\nu\left(\frac{1}{a}\right)$$

where $R(a)>0$, $\min\left(\frac{1}{4},\frac{\nu}{2}\right)>-2$.

$$(3{\cdot}5) \qquad \int_0^\infty x^{\frac{1}{2}}\, [H_0(ax)-Y_0(ax)]\, \varpi_{0,1}(x)\,dx = -\frac{1}{a^2}\left[H_{-1}\left(\frac{1}{a}\right)-Y_{-1}\left(\frac{1}{a}\right)\right]$$

where $|\arg a^2|<2\pi$.

$$(3{\cdot}6) \qquad \int_0^\infty x^{-\frac{1}{2}}\, H_{-\frac{1}{2}}(ax)\, \varpi_{\frac{1}{2},\frac{1}{2}}(x)\,dx = \sqrt{\frac{\pi}{2a^3}}\cos\frac{1}{a},\quad a>0.$$

$$(3{\cdot}7) \qquad \int_0^\infty x^{-\frac{1}{2}}\, H_{-\frac{1}{2}}(ax)\, \varpi_{\frac{1}{2},-\frac{1}{2}}(x)\,dx = \sqrt{\frac{\pi}{2a^3}}\sin\frac{1}{a},\quad a>0.$$

4. we shall derive the following sums now.

$$(4{\cdot}1) \qquad \sum_{r=0}^{\infty}(-1)^r(\mu+2r)\, b^{2r}\, J_{\frac{\mu+\nu}{2}+r}\left(\frac{1}{b}\right) = \frac{2^{(\mu-1)/2}}{b^{(3+\nu)/2}}\, J_{\frac{\mu+\nu}{2}-1}\left(\frac{1}{b}\right)$$

$$(4{\cdot}2) \qquad \sum_{r=0}^{\infty}(-1)^r(2r+1)\, b^{2r}\, J_{r+1}\left(\frac{1}{b}\right) = \left(\frac{1}{b^2}\right) J_0\left(\frac{1}{b}\right)$$

$$(4{\cdot}3) \qquad \sum_{r=0}^{\infty}(-1)^r(2r+1)\, I_{\frac{2r+1}{2}}\left(\frac{1}{2a}\right) K_{\frac{2r+1}{2}}\left(\frac{1}{2a}\right) = \frac{1}{4a^2}\, e^{-1/a}$$

$$(4\cdot4)\quad \sum_{r=0}^{\infty} (-1)^r (\mu+2r)\, G^{m,\, n+2}_{p+4,\, q}\left(16\lambda \left| \begin{matrix} \frac{1-\mu-2r}{2}-\rho,\ \frac{1-\nu}{2}-\rho,\ \alpha_i,\ \frac{1+\nu}{2}-\rho,\ \frac{1+\mu+2r}{2}-\rho \\ \beta_j \end{matrix}\right.\right)$$

$$= \frac{\lambda^{\rho-1}}{2^{4\rho+1}}\, G^{m,\, n+2}_{p+4,\, q}\left(16\lambda \left| \begin{matrix} 1+\alpha_1+\rho, \ldots, 1+\alpha_n+\rho, \frac{1-\nu}{2}, \frac{1-\mu}{2}, \frac{1+\mu}{2}, \frac{3+\nu}{2}, 1+\alpha_{n+1}+\rho, \ldots\ldots\ldots, 1+\alpha_p+\rho \\ 1+\beta_1+\rho, \ldots\ldots\ldots, 1+\beta_q+\rho \end{matrix}\right.\right)$$

Proof : Multiply both the sides of (2·1) by $x^{\frac{1}{2}(\nu-\mu+1)} J_{\frac{\mu+\nu}{2}}(bx)$ and then integrate it term by term with the help of (3·3) to deduce (4·1). Similarly other sums can be established.

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EFFECT OF A CONSTANT UNIFORM MAGNETIC FIELD ON THE SCATTERING OF ELECTROMAGNETIC WAVES BY FREE ELECTRON

By

JAYANTI DUTT, L. M. BALI

Physics Department, Lucknow University, Lucknow

and

VACHASPATI

Physics Department, Allahabad University, Allahabad

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ABSTRACT

The equation of motion of an electron in the presence of an external electromagnetic field and a constant uniform magnetic field is solved using perturbation method. It is shown that the scattered field consists of waves of fundamental frequency, the gyration frequency, their harmonics and their sum and difference frequencies. The effect of the applied magnetic field on the intensities of the scattered waves of fundamental frequency and its second harmonic is discussed. It is shown that the first and second harmonic cross sections become comparable when the incident frequency is equal to or half the gyration frequency of the electron. Variation of these cross sections with the direction of the applied magnetic field is also discussed.

INTRODUCTION

The equation of motion of an electron in a constant uniform magnetic field has been solved in many text books of electrodynamis[1-2]. The charge moves along a helix having its axis along the direction of the magnetic field. The angular frequency of rotation of the particle in a plane perpendicular to the field, $\vec{H}'$, is given by $\omega = eH'/mc$. The speed of the electron remains constant.

In this paper, we solve the equation of motion of an electron in a monochromatic beam of electromagnetic radiation, when a constant homogeneous magnetic field is applied in an arbitrary direction. We write the solution as a series in powers of the incident field strength, E_0, while the external magnetic field is treated exactly. The scattered field of the electron is evaluated and the effect of the applied magnetic field on the intensities of the scattered waves of fundamental incident frequency k_0 and its harmonic $2k_0$ are discussed. The electron oscillations of frequencies k_0 and $2k_0$ are superposed on the periodic motions due to the constant magnetic field; consequently, the scattered field contains waves of frequencies ω, 2ω, $\omega \pm k_0$ etc., besides those of k_0 and $2k_0$.

It is found, that the second harmonic cross section depends on the intensity and square of the wavelength of the incident beam as well as on the ratio of the gyration frequency to the incident frequency (k_0), and of the angle between the applied magnetic field and the direction of the incident beam (α). We find that its magnitude increases enormously near resonance at $\omega = k_0$, and $\omega = 2k_0$. Since the effect of the magnetic field is to superimpose a periodic motion on the original electron oscillations, resonant interaction between the electron and the

wave of a given frequency can be attained by suitably adjusting the external magnetic field. Thus if magnetic fields of required strength are available ($\simeq 10^8$ gauss for optical frequencies) the scattered radiation would contain harmonics which should be easily observable.

Application of a constant magnetic field is important from the point of dispersion also. A detailed calculation regarding the manner in which it affects the harmonic waves propagating inside a material medium is underway and would be presented in a subsequent publication.

1. EQUATION OF MOTION AND ITS SOLUTION

The equation of motion of an electron (mass $= m$, charge $= e$) in the presence of an external electromagnetic field, $F_{\alpha\mu}$, and a constant uniform magnetic field, $\vec{H}'$, can be written as

$$m \dot{V}_{\mu} = eV^{\alpha} \ F_{\alpha\mu} + e \ V^{\alpha} F'_{\alpha\mu}, \tag{1·1}$$

where $Z_{\mu} = (Z_0, Z_1, Z_2, Z_3)$ denotes the four coordinates of the particle and are functions of the proper time τ. $V_{\mu} = dZ_{\mu}/d\tau$, is the four velocity of the particle and

$$(F_{01}, F_{02}, F_{03}) \equiv \vec{E} = \vec{E}_0 \cos (kZ);$$

$$(F_{23}, F_{31}, F_{12}) \equiv \vec{H} = \vec{H}_0 \cos (kZ);$$

$$(F'_{01}, F'_{02}, F'_{03}) = 0; \quad (F'_{23}, F'_{31}, F'_{12}) = \vec{H}'.$$

we have also put the speed of light $c = 1$.

Making use of the series solution (4)

$$Z_{\mu} = \tau \ g_{0\mu} + \sum_{n=1}^{\infty} Z_{\mu}^{(n)},$$

(when n denotes the term containing E^{n}_0) in (1·1) and solving upto the second order in powers of E_0, we get

$$\begin{aligned}\vec{Z} = &- (qP_1/k_0) \left\{ \vec{S} \cos k_0\tau - g \, [\vec{e}_0, \vec{i}_1] \sin k_0 \tau \right\} + \vec{R}^{(1)}_{2+3} \cos (\omega\tau + A^{(1)}_{2+3}) \\ &+ (q^2P_1P_2/8k_0) \left\{ \vec{E}_s \sin 2k_0 \tau + \vec{E}_c \cos 2k_0 \tau \right\} \\ &+ (qR^{(1)}/2) \sum_{a=\pm} \left\{ \vec{M}^{c}_a \cos ((\omega+ak_0) \tau + A^{(1)}) + \vec{M}^{s}_a \sin ((\omega+ak_0)\tau+A^{(1)}) \right\} \\ &+ \vec{R}^{(2)}_{2+3} \cos (\omega\tau + A^{(2)}_{2+3}),\end{aligned} \tag{1·2,a}$$

and

$$\begin{aligned}Z_0 = &(1 + B_0^{(2)}) \tau - (q^2P_1/8k_0) (\vec{S}. \vec{e}_0) \sin 2k_0 \tau \\ &+ \sum_{a=\pm} (qR^{(1)} g/2) (1 + ag)^{-2} \left\{ (\vec{e}_0. \vec{i}_2) \sin ((\omega + ak_0) \tau + A^{(1)}) \right. \\ &\left. + (\vec{e}_0. \vec{i}_3) \cos ((\omega + ak_0) \tau + A^{(1)}) \right\}.\end{aligned} \tag{1·2,b}$$

Here we have used the coordinate system in which the electron is on the average at rest at the origin, and the zero of the electron's time coincides with the zero of the proper time τ.

The constant $B_0^{(2)}$ can be obtained by making use of the relation

$$V_\mu V^\mu = 1$$

$R^{(n)}$ ($n = 1, 2$) are related to the radius of the circular motion of frequency, ω, in a plane perpendicular to the magnetic field while $A^{(n)}$ occur as phase factors. These quantities may be determined by fixing the time at which the external magnetic field is switched on. However, since they do not occur in our final expression we do not use their explicit values.

In (1·2) we have put

$$q = (eE_0/mk_0); \quad \vec{e}_0 = \vec{E}_0/E_0, \quad \vec{n}_0 = \vec{k}/k_0, \quad |\vec{E}_0| = |\vec{H}_0| = E_0,$$

$$\omega = eH'/m, \quad g = \omega / k_0, \quad P_n = (g^2 - n^2)^{-1}$$

$$\vec{R}^{(n)}_{2+3} \cos (\omega \tau + A^{(n)}_{2+3}) \equiv \vec{R}^{(n)}_2 \cos (\omega \tau + A^{(n)}_2) + \vec{R}^{(n)}_3 \cos (\omega \tau + A^{(n)}_3)$$

with

$$\vec{R}^{(n)}_2 = R^{(n)} \vec{i}_2 ; \quad \vec{R}^{(n)}_3 = R^{(n)} \vec{i}_3 ; \quad A^{(n)}_3 = A^{(n)}_2 + \pi/2.$$

$$\vec{S} = \left[g^2 (\vec{e}_0 \vec{i}_1) \vec{i}_1 - \vec{e}_0 \right],$$

$$\vec{E}_S = \Big\{ g^2 ((\vec{S} \cdot \vec{e}_0) + 2) (\vec{n}_0 \vec{i}_1) \vec{i}_1 - 8g^2 (\vec{e}_0 \vec{i}_1) (\vec{n}_0 \vec{i}_1) \vec{e}_0$$
$$- 4 g^2 ([\vec{n}_0 \, \vec{e}_0] \vec{i}_1) [\vec{e}_0 , \vec{i}_1] + 4 (\vec{S} \cdot \vec{e}_0) \vec{n}_0 \Big\},$$

$$\vec{E}_C = (g/2) [\vec{i}_1 , \vec{E}_S] + (2g/P_2) ([\vec{n}_0 , \vec{e}_0] \vec{i}_1) \vec{e}_0 ,$$

$$\vec{M}^c_a = a (1 + ag)^{-2} (1 + 2ag)^{-1} \Big[ag (1 + 2ag) (\vec{e}_0 \vec{i}_2) \vec{n}_0$$
$$- \Big\{ (\vec{n}_0 \vec{i}_3) \vec{e}_0 - ag ([\vec{n}_0 , \vec{e}_0] \vec{i}_1) \vec{i}_2 \Big\} (1 + 2ag + 2g^2)$$
$$+ ag (\vec{e}_0 \vec{i}_1) [\vec{n}_0 , \vec{i}_2] + g^2 (\vec{e}_0 \vec{i}_1) (\vec{n}_0 \vec{i}_1) \vec{i}_3 \Big]$$

$$\vec{M}^s_a = a (1 + ag)^{-2} (1 + 2ag)^{-1} \Big[ag (1 + 2ag) (\vec{e}_0 \vec{i}_2) \vec{n}_0$$
$$- \Big\{ (\vec{n}_0 \vec{i}_2) \vec{e}_0 + ag ([\vec{n}_0 , \vec{e}_0] \vec{i}_1) \vec{i}_3 \Big\} (1 + 2ag + 2g^2)$$
$$- ag (\vec{e}_0 \vec{i}_1) [\vec{n}_0 , \vec{i}_3] + g^2 (\vec{e}_0 \vec{i}_1) (\vec{n}_0 \vec{i}_1) \vec{i}_2 \Big].$$

Here $\vec{i}_1$ is the unit vector in the direction of the applied magnetic field, and $\vec{\imath}_2$ and $\vec{\imath}_3$ are two mutually perpendicular unit vectors which, along with $\vec{i}_1$, from a right handed orthogonal coordinate system.

II. EVALUATION OF THE SCATTERED FIELD

The electric and magnetic field of the scattered wave at a space time point $(\vec{x}, x_0)$ produced as a result of the electron motion (1·2, *a–b*), are given by

$$\vec{H}^{sc} = e/R \left[\vec{n}, \vec{M} \right]; \vec{E}^{sc} = \left[\vec{H}^{sc}, \vec{n} \right] \tag{2·1}$$

where

$$\vec{M} = \left[V_0 - (\vec{n}\,\vec{V}) \right]^{-3} \left[(\dot{V}_0 - (\vec{n}\,\dot{\vec{V}}))\,\vec{V} - (V_0 - (\vec{n}\,\vec{V}))\,\dot{\vec{V}} \right. \tag{2·2}$$

$$\vec{R} = (\vec{x} - \vec{z}); \vec{n} = \vec{x} - \vec{z} / / \vec{R} / .$$

In order to calculate the scattering cross sections for waves of fundamental frequency k_0 and its harmonic $2k_0$, in their lowest orders, we must find out the scattered field upto the second order in the strength of external electromagnetic radiation. Differentiating equation (1·2, *a – b*) and substituting in (2·2) we get $\vec{M}$ as a function of the retarded proper time, τ_r, at which

$$x_0 - z_0(\tau_r) = / \vec{x} - \vec{z}(\tau_r) / \approx r - (\vec{n}.\vec{z}); / \vec{x} / = r \tag{2·3}$$

we may express τ_r as a function of the coordinates of the point of observation by solving (1·2, *b*) for τ. We get

$$k_0 \tau_r = \phi^{(0)} + \phi^{(1)} + ---$$

where

$$\phi^{(0)} \equiv \phi = k_0 (x_0 - r),$$

$$\phi^{(1)} = qP_1 \left[g(\vec{n} [\vec{e}_0, \vec{i}_1]) \sin \phi - (\vec{S}.\vec{n}) \cos \phi \right]$$

$$+ k_0 (\vec{R}^{(1)}_{2+3}\, n) \cos (\chi + A^{(1)}_{2+3})$$

and

$$\chi = g\,\phi$$

The scatterd field can now be written as

$$\vec{M} = \vec{N}^{(1)} + \vec{N}^{(2)} + ---$$

where

$$\vec{N}^{(1)} = -qk_0P_1\left[\vec{S}\cos\phi - g\,[\vec{e}_0, \vec{i}_1]\sin\phi\right] + \omega^2\,\vec{R}^{(1)}_{2+3}\cos(\chi + A^{(1)}_{2+3}). \quad (2\cdot4, a)$$

$$\begin{aligned}\vec{N}^{(2)} &= (q^2k_0P_1/2)\left[(P_2\vec{E}_s - 4P_1\vec{F}_s)\sin 2\phi + (P_2\vec{E}_c - 4P_1\vec{F}_c)\cos 2\phi\right]\\
&+ \sum_{a=\pm}(qk_0^2R^{(1)}/2)(1+ag)^2\Bigg[\Big\{\vec{M}^c_a + P_1\,(g(\vec{s}\,\vec{n})\vec{i}_3\\
&+ ag^2(\vec{n}[\vec{e}_0, \vec{i}_1])\vec{i}_2 + a(\vec{n}\,\vec{i}_3)\vec{S}\\
&+ g(\vec{n}\,\vec{i}_2)[\vec{e}_0, \vec{i}_1])\Big\}\cos(\chi + a\phi + A^{(1)})\\
&+ \Big\{\vec{M}^s_a + P_1\,(g(\vec{s}\,\vec{n})\,\vec{i}_2 - ag^2(\vec{n}[\vec{e}_0, \vec{i}_1])\vec{i}_3\\
&+ a(\vec{n}\,\vec{i}_2)\vec{S} - g(\vec{n}\,\vec{i}_3)[\vec{e}_0, \vec{i}_1])\Big\}\sin(\chi + a\phi + A^{(1)})\Bigg]\\
&- 2\omega^3 R^{(1)\,2}\Big[\Big\{(\vec{n}\,\vec{i}_2)\vec{i}_2 - (\vec{n}\,\vec{i}_3)\vec{i}_3\Big\}\,.\sin(2\chi + 2A^{(1)})\\
&+ \Big\{(\vec{n}\,\vec{i}_3)\vec{i}_2 + (\vec{n}\,\vec{i}_2)\vec{i}_3\Big\}\,.\cos(2\chi + 2A^{(1)})\Big]\\
&+ \omega^2\,\vec{R}^{(1)}_{2+3}\,.\cos(\chi + A^{(2)}_{2+3})\,. \end{aligned} \quad (2\cdot4, b)$$

In the first approximation it consists of waves of fundamental frequency k_0 and the gyration frequency ω. In the second order it contains the second harmonic of frequencies k_0 and ω and the beat frequencies $\omega \pm k_0$. Higher harmonics and their sum and difference frequencies would arise when subsequent orders are taken into account.

III. EVALUATION OF THE SCATTERING CROSS SECTION

The differential scattering cross section $d\sigma$, for scattering of the radiation into a solid angle $d\Omega$ is given by

$$(d\sigma / d\Omega) = (2e^2 / E_0^2)\,[\vec{n}, \vec{M}]^2_{av} \quad (3\cdot1)$$

Using (2 4, a–b), (3·1) can be written as the sum of scattering cross-sections for waves of different frequencies k_0, ω, $2k_0$, 2ω and $\omega \pm k_0$. For unpolarised light, the differential scattering cross-sections for waves of fundamental frequency k_0 and its harmonic $2k_0$ are*

*When calculations are done upto an order sufficient to give σ_{2k_0} in its lowest order, σ_{k_0} would consist of two parts ; the intensity independent part given by (3·2a) and an intensity dependent part. Since the latter is a second order effect which at the most can become of the order of the intensity independent part we have discussed only the more significant part given by (3·2a).

$$\left(\frac{d\sigma k_0}{d\Omega}\right)_{\text{unpol}} = \tfrac{1}{2}P_1^2\cdot(e^2/m)^2\Big[2(1+g^2(\vec{n}_0\,\vec{i}_1)^2)$$

$$+(g^2-1)(1-(\vec{n}_0\,\vec{n})^2)+g^2(g^2-g^2(\vec{n}_0\,\vec{i}_1)^2-2)(1-(\vec{n}\,\vec{i}_1)^2)\Big]; \quad (3{\cdot}2,a)$$

$$\left(\frac{d\sigma 2k_0}{d\Omega}\right)_{\text{unpol}} = (e^2/m)^2 g^2 P_1^2\Big[X_1^2-2X_1X_2+X_2^2\Big]_{\text{unpol}} \quad (3{\cdot}2,b)$$

where

$$(X_1^2/P_2^2)_{\text{unpol}} = (g^2/32)\Big[8(g^4-4g^2+36)-g^2(3g^4-80g^2+168)$$

$$\times(\vec{n}_0\,\vec{i}_1)^2+2g^4(3g^2-28)(\vec{n}_0\,\vec{i}_1)^4-3g^6(\vec{n}_0\,\vec{i}_1)^6\Big](\vec{n}\,\vec{i}_1)^2$$

$$-(g^2/4)\Big[(7g^4-48g^2+80)+g^2(g^2-4)(\vec{n}_0\,\vec{i}_1)^2\Big]$$

$$(\vec{n}_0\,\vec{i}_1)(\vec{n}_0\,\vec{n})(\vec{n}\,\vec{i}_1)+(1/32)\Big[\Big\{4(g^6+8g^4+16g^2+32)$$

$$+g^2(3g^6-8g^4+120g^2-128)(\vec{n}_0\,\vec{i}_1)^2-2g^4(3g^4-14g^2+72)$$

$$\times(\vec{n}_0\,\vec{i}_1)^4+3g^6(g^2-8)(\vec{n}_0\,\vec{i}_1)^6\Big\}+\Big\{4(3g^6-32g^4+88g^2+32)$$

$$+8g^2(g^4-6g^2+8)(\vec{n}_0\,\vec{i}_1)^2+12g^4(g^2-4)(\vec{n}_0\,\vec{i}_1)^4\Big\}(\vec{n}_0\,\vec{n})^2\Big] \quad (3{\cdot}3,a)$$

$$\left(\frac{X_1X_2}{P_1P_2}\right)_{\text{unpol}} = -(g^4/8)\Big[(3g^4-38g^2+74)$$

$$-6g^2(g^2-4)(\vec{n}_0\,\vec{i}_1)^2+3g^4(\vec{n}_0\,\vec{i}_1)^4\Big](\vec{n}_0\,\vec{i}_1)(\vec{n}\,\vec{i}_1)^3$$

$$-(g^4/2)\Big[3(g^2-4)+(4g^2-19)(\vec{n}_0\,\vec{i}_1)^2\Big](\vec{n}_0\,\vec{n})(\vec{n}_0\,\vec{i}_1)^2$$

$$+(g^2/8)\Big[\Big\{(3g^6-18g^4+92g^2+148)-6g^2(g^4-5g^2+8)(\vec{n}_0\,\vec{i}_1)^2$$

$$+3g^4(g^2-4)(\vec{n}_0\,\vec{i}_1)^4\Big\}+\Big\{(5g^4-15g^2-32)$$

$$+9g^2(g^2-3)(\vec{n}_0\,\vec{i}_1)^2\Big\}(\vec{n}_0\,\vec{n})^2\Big](\vec{n}_0\,\vec{i}_1)(\vec{n}\,\vec{i}_1)$$

$$+1/8\Big[\Big\{(2g^6-16g^2+16)+g^2(g^4+3g^2+8)(\vec{n}_0\,\vec{i}_1)^2$$

$$-3g^4(g^2-7)(\vec{n}_0\,\vec{i}_1)^4\Big\}-\Big\{(12g^4-40g^2+16)$$

$$-12g^2(\vec{n}_0\,\vec{i}_1)^2\Big\}(\vec{n}_0\,\vec{n})^2\Big](\vec{n}_0\,\vec{n}), \quad (3{\cdot}3,b)$$

$$\left(\frac{X_2^2}{P_1^2}\right)_{\text{unpol}} = -(g^4/8)\Big[3(g^4 \quad 4g^2+4)$$

$$-2g^2(3g^2-4)(\vec{n}_0\vec{i}_1)^2+3g^4(\vec{n}_0\vec{i}_1)^4\Big](\vec{n}\vec{i}_1)^4$$

$$+(g^2/2)(g^2-1)\Big[(g^2-1)-2g^2(\vec{n}\vec{i}_1)^2\Big](\vec{n}_0\vec{i}_1)(\vec{n}_0\vec{n})(\vec{n}\vec{i}_1)$$

$$+(g^2/8)(g^2-1)\Big[\Big\{(3g^4-8g^2+2)-6g^2(g^2-1)(\vec{n}_0\vec{i}_1)^2+3g^4(\vec{n}_0\vec{i}_1)^4\Big\}$$

$$-\Big\{(2g^2-8)-6g^2(\vec{n}_0\vec{i}_1)^2\Big\}(\vec{n}_0\vec{n})^2\Big](\vec{n}\vec{i}_1)^2$$

$$+1/8\Big[\Big\{(g^4+g^2+1)+g^2(3g^4-6g^2+5)(\vec{n}_0\vec{i}_1)^2$$

$$-3g^4(g^2-2)(\vec{n}_0\vec{i}_1)^4\Big\}+(g^6-3g^2+2)-3g^2(g^2-1)^2(\vec{n}_0\vec{i}_1)^2\Big\}(\vec{n}_0\vec{n})^2$$

$$-3(g^2-1)^2(\vec{n}_0\vec{n})^4\Big] \qquad (3{\cdot}3,\ c)$$

These cross-sections are evaluated for $\omega \neq k_0$ and $\omega \neq 2k_0$, since due to the neglect of radiation damping, they become infinite at resonance.

The scattered field also contains waves of frequencies other than k_0 and $2k_0$. As can be seen from equations (3·2, a) – (3·3, c) the angular dependence is quite complicated and involves not only the direction of observation relative to the direction of the incident field but also the direction of the applied magnetic field and various powers of the ratio ω/k_0. Relative improtance of these in thc scattered field can therefore be obtained for cases of special interest for frequencies k_0 and $2k_0$.

IV. TOTAL CROSS-SECTION

Total cross-sections for scattering into the waves of first and harmonic frequencies can be obtained by integrating over the solid angle $d\Omega = \sin\psi\, d\psi\, d\phi$, and writing.

$$\cos\delta = \cos\alpha\cos\psi + \sin\alpha\sin\psi\cos\phi$$

where we have defined

$$\cos\delta = (\vec{n}_0\vec{n})\,;$$

$$\cos\psi = (\vec{n}\vec{i}_1)\,;$$

$$\cos\alpha = (\vec{n}_0\vec{i}_1)\,.$$

We obtain

$$\sigma_{k_0} = \tfrac{1}{2}\left[\sin^2\alpha + (2-\sin^2\alpha)\cdot\frac{(g^2+1)}{(g^2-1)^2}\right]\sigma_T \tag{4·1}$$

$$\begin{aligned}\sigma_{2k_0} = (1/160\, q^2)\, P_1^2 \Big[& 5\, P_2^2 \{ 8 (g^4 - 8 g^2 + 16) \\ & + 8 g^2 (9 g^4 + 23 g^2 + 60) \sin^2\alpha \\ & + g^4 (3 g^4 - 92 g^2 - 224) \sin^4\alpha - 3 g^6 (g^2 - 12) \sin^6\alpha \} \\ & + 16 P_1^2 \{ 8 (g^4 + 4 g^2 + 1) + 16 g^2 (g^4 - 2 g^2 - 2) \sin^2\alpha \\ & + 3 g^4 (g^4 - 8 g^2 + 13) \sin^4\alpha \} \Big]\end{aligned} \tag{4·2}$$

Where, σ_T, is the Thomson scattering cross-section.

For $g = 0$, these expressions reduce to those for free electron at rest.

DISCUSSION

From (4·1) the following points are to be noted :

(*i*) σ_{k_0} in its lowest order is independent of the intensity of the incident radiation. It however depends on the angle, α, between the applied magnetic field and the direction of the incident light, and on the ratio $g = \omega/k_0$, of the gyration frequency to the incident frequency.

(*ii*) For fixed α, σ_{k_0} varies with g. It depends on the function $T = (g^2+1)/(g^2-1)^2$

(*a*) If $T < 1$ *i.e.* $g^2 > 3$, $\sigma_{k_0} < \sigma_T$

(*b*) If $T = 1$ *i.e.* $g^2 = 0$, $\sigma_{k_0} = \sigma_T$

(*c*) If $T > 1$ *i.e.* $3 > g^2 > 0$, $\sigma_{k_0} > \sigma_T$

(*iii*) For fixed g, σ_{k_0} varies with α. It has a maximum at $\alpha = 10°$ for $g^2 < 3$; at $\alpha = 90°$ for $g^2 > 3$, and is independent of α at $g^2 = 3$.

(*iv*) At resonance the cross-section becomes infinite due to the neglect of radiation damping. To get an idea of its magnitude in regions close to resonance, we take $(g^2 - 1)^{-2} \sim 1/\gamma^2 k_0^2$ and $\alpha = 0°$.

where $\gamma = 2\, e^2/3m$ is the damping constant. We then get

$$\sigma_{k_0} \sim 2 \times 10^{16}\ \sigma_T. \quad (\text{near } g \approx 1)$$

For $(g^2 - 1)^{-2} = \frac{1}{9}$ and $\alpha = 54°$, we get

$$\sigma_{k_0} \sim 0{\cdot}55\ \sigma_T. \quad (\text{near } g \approx 2)$$

From (4·2), we see

(*i*) In its lowest order σ_{2k_0} depends on the intensity and square of the wave length of the incident beam as well as on g and α.

(*ii*) For fixed values of α, we can write

$$\sigma_{2k_0} = q^2 \ \sigma_T \left[K^{(1)} + K^{(2)} P_1 + K^{(3)} P_1^2 + K^{(4)} P_1^2 P_2 + K^{(5)} P_1^2 P_2^2 + K^{(6)} g^2 P_1^4 \right],$$

where $P_n = (g^2 - n^2)^{-1}$, and coefficients $K^{(n)}$ depend on the particular choice of the direction of the applied magnetic field. In general it will have two maxima, at $g = 1$ and $g = 2$, but the coefficients $K^{(n)}$ are such that for $\alpha = 0°$ it only has one maximum at $g = 1$.

(*iii*) For fixed g, σ_{2k_0} varies with α and is maximum at $\alpha = 0°$ in regions close to $g = 1$, and at about $\alpha \sim 54°$ in regions close to $g = 2$. Maximum at $g \approx 1$, $\alpha = 0°$ will be more pronounced than at $g \approx 2$, $\alpha = 54°$.

(*iv*) At resonance σ_{2k_0} becomes infinite. We therefore calculate its value in regions close to $g = 1$ and $g = 2$ respectively, by taking

(*a*) $(g^2 - 1)^{-2} \sim 1/\gamma^2 k_0^2$ and $\alpha = 0°$, which gives

$$\sigma_{2k_0} \sim 3 \times 10^{49} q^2 \lambda^4 \ \sigma_T \text{ (near first resonance)}$$

(*b*) $(g^2 - 4)^{-2} \approx 1/\gamma^2 k_0^2$ and $\alpha = 54°$. This gives

$$\sigma_{2k_0} \approx 3 \times 10^{25} \cdot q^2 \ \lambda^2 \ \sigma_T \text{ (near the second resonance)}$$

Writing $q^2 = e^2 E_0^2/m^2 k_0^2 = (2 e^2/\pi m^2) \lambda^2 I_0$,

With $I_0 = E_0^2/8\pi$ as the incident energy flux and taking λ, in the optical region $\sim 7 \times 10^{-5}$ cm we find in the above two cases that

(*a*) $\sigma_{2k_0} \sim 2 \times 10^6 \ I_0 \ \sigma_T$ and $\sigma_{k_0} \approx 2 \times 10^{16} \ \sigma_T$ near $g = 1$)

(*b*) $\sigma_{2k_0} \sim 0{\cdot}5 \times 10^{-8} I_0 \ \sigma_T$, and $\sigma_{k_0} \sim 0{\cdot}55 \ \sigma_T$ (near $g = 2$)

We thus see that σ_{2k_0} becomes comparable to σ_{k_0} even with ordinary sources for which $I_0 \approx 10^9$ ergs/cm² sec. The cross section $2k_0$ in case (*a*) is much greater than in the case (*b*). The magnitudes given here are not exact, since the slight variations that would occur in the expressions for σ_{k_0} and σ_{2k_0} at resonance have not been taken into account. Nevertheless, this gives us an idea of the enormous increase in σ_{2k_0} and σ_{k_0} near resonance.

Enormous increase in the scattering cross sections at resonance have been pointed out previously in the scattering of light by bound electrons[5]. In order to attain resonance we have to choose a particular material whose natural frequency is the same as the frequency of the incident light. But in the present case resonance for a given incident frequency can be attained simply by adjusting the external magnetic field. However, very strong fields $\sim 10^8$ gauss are required to excite resonance for optical incident frequencies.

A. A. Kolomenskii and A. N. Lebedev[3] have discussed the motion of an electron under the impressed electromagnetic field, when a constant uniform magnetic field is applied in the direction of the incident beam. They have shown that resonance can be maintained automatically despite the increase in the energy of the particle and corresponding decrease in frequency.

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SOME FINITE AND INFINITE INTEGRALS INVOLVING G-FUNCTION

By

P. K. SUNDARARAJAN

Department of Mathematics, S. F. S. College, Nagpur

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ABSTRACT

In this paper we prove a theorem on Stieltjes transform and make use of that to evaluate an infinite integral involving G-function whose argument is a power of the variable. We also make use of a theorem by Harishanker [6] to evaluate some finite integrals involving G-function.

1. INTRODUCTION

The Laplace and generalized Stieltjes transforms are defined by the integral equations

$$h(s) = s\int_0^\infty e^{-st} f(t)\, dt \tag{1·1}$$

$$\varphi(p) = p\int_0^\infty (p + x)^{-\lambda} f(x)\, dx \tag{1·2}$$

respectively. In this note we shall denote (1·1) and (1·2) symbolically as

$$f(t) \doteqdot h(s) \text{ and } f(x) \underset{\lambda}{\overset{S}{\rightarrow}} \varphi(p)$$

2. *Theorem* : If $f(x) \underset{\lambda}{\overset{S}{\rightarrow}} g(p)$ and $x^{\nu-2}\, g(x) \underset{\mu}{\rightarrow} \varphi(p)$

then

$$\varphi(p) = \frac{\Gamma(\nu)\,\Gamma(\mu - \nu + \lambda)\, p^{\nu-\mu+1}}{\Gamma(\mu + \lambda)} \int_0^\infty t^{-\lambda}\, {}_2F_1\left(\lambda, \nu : \mu + \lambda : 1 - \frac{p}{t}\right) f(t)\, dt.$$

provided $R(\nu) > 0$, $R(\mu) > R(\nu - \lambda)$, $f(x)$ is continuous and non-negative in $(0, \infty)$, $R(p) > 0$, and the above integral converges.

Proof. By hypothesis

$$g(p) = p\int_0^\infty (p + t)^{-\lambda} f(t)\, dt \tag{2·1}$$

and

$$\varphi(p) = p\int_0^\infty (p + x)^{-\mu}\, x^{\nu-2}\, g(x)\, dx. \tag{2·2}$$

Substituting the value of $g(x)$ from (2·1) in (2·2) and inverting the order of integrations we get

$$\varphi(p) = p\int_0^\infty f(t) \left[\int_0^\infty (p + x)^{-\mu}\, x^{\nu-1}\, (t + x)^{-\lambda}\, dx\right] dt \tag{2·3}$$

Evaluating the x integral using [4, p. 233, 14·4 (9)] we get

$$(2\cdot4)\quad \varphi(p) = \frac{\Gamma(\nu)\,\Gamma(\mu-\nu+\lambda)}{\Gamma(\mu+\lambda)}\, p^{\nu-\mu+1} \int_0^\infty t^{-\lambda}\, {}_2F_1\left(\lambda, \nu : \mu + \lambda : 1-\frac{p}{t}\right) f(t)\, dt$$

provided $R(\nu) > 0$, $R(\mu) > R(\nu-\lambda)$.

The change of order of integrations can be justified as follows :

In (2·3) the x integral is convergent provided $R(\nu) > 0$, $R(\mu) > R(\nu-\lambda)$. The t integral converges by virtue of (2·1). Also the repeated integral is convergent by hypothesis. The integral is non-negative and so by de la Valle Poussin's theorem [1, p. 504] the change of order of integrations is justified.

Example. Consider [8]

$$(2\cdot5)\qquad f(x) = x^{\rho-1}\, G^{l,n}_{u,v}\left(x^r \beta \,\middle|\, \begin{matrix} a_1, \ldots, a_u \\ b_1, \ldots, b_v \end{matrix}\right) \xrightarrow[\lambda]{s}$$

$$\frac{r^{\lambda-1}}{\Gamma(\lambda)\,(2\pi)^{r-1}}\, p^{\rho-\lambda+1}\, G^{l+r,\, n+r}_{u+r,\, v+r}\left(p^r \beta \,\middle|\, \begin{matrix} \varphi_0, \ldots, \varphi_{r-1}, a_1, \ldots, a_u \\ \psi_0, \ldots, \psi_{r-1}, b_1, \ldots, b_v \end{matrix}\right) = g(p)$$

where $\phi_R = \dfrac{r-\rho-R}{r}$, $\psi_R = \dfrac{\lambda-\rho+R}{r}$, $R = 0, 1, \ldots, r-1$.

$R\{r(1-a_j) + \lambda - \rho\} > 0, j = 1, 2, \ldots n.$

$R(rb_h + \rho) > 0$, $h = 1, 2, \ldots, l$. $|\arg \beta| < (l + n - \frac{1}{2}u - \frac{1}{2}v)\,\pi$ $|\arg p| < \pi$, $u + \nu < 2(1 + n)$, and r is a positive integer.

From (2·5) we can easily show

$$(2\cdot6)\qquad x^{\nu-2}\, g(x) \xrightarrow[\mu]{s} \frac{r^{\mu+\lambda-2}}{\Gamma(\lambda)\,\Gamma(\mu)\,(2\pi)^{2r-2}}\, p^{\nu+\rho-\lambda-\mu+1}$$

$$\times\ G^{l+2r,\, n+2r}_{u+2r,\, v+2r}\left(p^r \beta \,\middle|\, \begin{matrix} \varphi_0, \ldots, \varphi_{r-1}, X_0, \ldots, X_{r-1}, a_1, \ldots, a_u \\ \psi_0, \ldots, \psi_{r-1}, Y_0, \ldots, Y_{r-1}, b_1, \ldots, b_v \end{matrix}\right) = \varphi(p)$$

where $X_R = \dfrac{r-\nu-\rho+\lambda-R}{r}$, $Y_R = \dfrac{\mu-\nu-\rho+\lambda+R}{r}$

$R = 0, 1, \ldots, r-1$. ϕ_R and ψ_R as defined in (2·4) provided

$R(r\, b_h + \nu + \rho - \lambda) > 0$, $h = 1, 2, \ldots, 1 - r$.

$R\{r(1-a_j) + \lambda - \rho\} > 0$, $j = 1, 2, \ldots, n-r$.

$R(\nu) > 0$, $R(\lambda) > 0$, $|\arg \beta| < (l + n - \frac{1}{2}u - \frac{1}{2}v)\pi$.

$|\arg p| < \pi$, $u + v < 2(1 + n)$.

Hence applying (2·4) we get

$$\int_0^\infty t^{\rho-\lambda-1}\, G^{l,n}_{u,v}\left(t^r\,\beta\,\middle|\,\begin{matrix}a_1,\ldots,a_u\\ b_1,\ldots,b_v\end{matrix}\right)\, {}_2F_1\left(\lambda,\nu:\mu+\lambda:1-\frac{p}{t}\right)dt$$

$$=\frac{\Gamma(u+\lambda)\; r^{\mu+\lambda-2}\; p^{\rho-\lambda}}{\Gamma(\lambda)\,\Gamma(\mu)\,\Gamma(\nu)\,\Gamma(\mu-\nu+\lambda)\,(2\pi)^{2r-2}}$$

$$\times\, G^{l+2r,\;n+2r}_{u+2r,\;v+2r}\left(p^r\,\beta\,\middle|\,\begin{matrix}\varphi_0,\ldots,\varphi_{r-1},X_0,\ldots,X_{r-1},a_1,\ldots,a_u\\ \psi_0,\ldots,\psi_{r-1},\Upsilon_0,\ldots,\Upsilon_{r-1},b_1,\ldots,b_v\end{matrix}\right)$$

where the parameters and the conditions are as in (2·5) and (2·6), provided $R(\mu)>R(\nu-\lambda)$.

3. In this section we can establish the result

$$(3\cdot1)\qquad \int_0^\infty (t+a)^{\sigma-1}\,(t^2+2at)^{-\lambda/2}\;P^{\lambda}_{\mu}\left(1+\frac{t}{a}\right)$$

$$G^{l,n}_{p,q}\left\{\frac{\beta}{(t+a)^{2r}}\,\middle|\,\begin{matrix}a,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right\}dt$$

$$=a^{\sigma-\lambda}\,(2r)^{\lambda-1}\;G^{l,n+2r}_{p+2r,\;q+2r}\left(\frac{\beta}{a^{2r}}\,\middle|\,\begin{matrix}\theta_0,\ldots,\theta_{r-1},\alpha_0,\ldots,\alpha_{r-1},a_1,\ldots,a_p\\ b_1,\ldots,b_q,\varphi_0,\ldots,\varphi_{2r-1}\end{matrix}\right)$$

where $r\theta_R=\frac{1}{2}(\sigma-\lambda-\mu-1)+r-R$, $r\alpha_R=\frac{1}{2}(\sigma-\lambda+\mu)+r-R$, $R=0,1,\ldots,r-1$

and $$\varphi_R=\frac{\sigma+2r-1-R}{2r},\; R=0,1,\ldots,2r-1.$$

$|\arg a|<\pi$, $R(2rb_h-\sigma+1)>0$, $h=1,2,\ldots,l.$

$R(2rb_h-\sigma+\lambda\pm\mu)>0$, $h=1,2,\ldots,l.$ $R(a)>0$,

$p+q<2(1+n)$, $R(\lambda)<1$, and r is a positive integer.

In proving the above formula we shall require the following result[8]

$$(3\cdot2)\qquad \int_0^\infty x^{-\rho}\,K_{2m}(2\sqrt{ax})\;G^{l,n}_{p,q}\left(x^r y\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)dx$$

$$=\frac{a^{\rho-1}}{2(2\pi)^{r-1}\,r^{2\rho-1}}\;G^{l,\,n+2r}_{p+2r,\,q}\left(\frac{r^{2r}y}{a^r}\,\middle|\,\begin{matrix}\theta_0,\ldots,\theta_{r-1},\alpha_0,\ldots\alpha_{r-1},a_1,\ldots a_p\\ b_1,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,b_q\end{matrix}\right)$$

where $$\theta_R=\frac{\rho+r-m-1-R}{r},\;\alpha_R=\frac{\rho+r+m-1-R}{r}$$

$R=0,1,\ldots,r-1$ and r is a positive integer provided

$$p+q<2(l+n),\;\left|\arg\frac{y}{a^r}\right|<(l+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\,\pi$$

$R(rb_h-\rho+1\pm m)>0$, $h=1,2,\ldots,l.$ $R(a)>0.$

Proof. Now consider the operational relations [3, p. 278] and [7]

$$(3\cdot3)\qquad f_1(t) = \left(\frac{\pi}{2a}\right)^{\frac{1}{2}} (t^2 + 2at)^{-\frac{\lambda}{2}} P_{\mu}^{\lambda}\left(1 + \frac{t}{a}\right)$$

$$\doteqdot s^{\lambda+\frac{1}{2}} e^{as} K_{\mu+\frac{1}{2}} (as) = g_1(s)$$

$R(\lambda) < 1$, $|\arg a| < \pi$, and

$$(3\cdot4)\qquad f_2(t) = e^{-at} t^{-\sigma} G_{p,q+2r}^{l,n}\left(t^{2r} y \,\middle|\, \begin{matrix} a_1, \ldots\ldots\ldots\ldots, a_p \\ b_1 \ldots b_q, \varphi_0, \ldots \varphi_{2r-1} \end{matrix}\right)$$

$$\doteqdot \frac{s(s+a)^{\sigma-1}}{(2\pi)^{(2r-1)/2}(2r)^{\sigma-1/2}} G_{p,q}^{l,n}\left\{\frac{(2r)^{2r} y}{(s+a)^{2r}} \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right\} = g_2(s)$$

where $\varphi_R = \dfrac{\sigma+2r-1-R}{2r}$, $R = 0, 1, \ldots, 2r-1$.

$R(2rb_h - \sigma + 1) > 0$, $h = 1, 2, \ldots, l$. $p + q < 2(l + n)$,

$\left|\arg \dfrac{s^r}{y}\right| < \left(l+n+\dfrac{r}{2}-\dfrac{p}{2}-\dfrac{q}{2}\right)\pi$, and r is a positive integer.

Applying Goldstein's theorem [5] to the above operational relations we get

$$(3\cdot5)\qquad \int_0^{\infty} (t+a)^{\sigma-1} (t^2+2at)^{-\lambda/2} P_{\mu}^{\lambda}\left(1+\frac{t}{a}\right) G_{p,q}^{l,n}\left\{\frac{(2r)^{2r} y}{(t+a)^{2r}} \,\middle|\, \begin{matrix} a_1, \ldots a_p \\ b_1, \ldots b_q \end{matrix}\right\} dt$$

$$= \left(\frac{2a}{\pi}\right)^{\frac{1}{2}} (2\pi)^{\frac{1}{2}(2r-1)} (2r)^{\sigma-\frac{1}{2}}$$

$$\times \int_0^{\infty} t^{-\sigma+\lambda-\frac{1}{2}} K_{\mu+\frac{1}{2}}(at)\, G_{p,q+2r}^{l,n}\left(t^{2r} y \,\middle|\, \begin{matrix} a_1, \ldots\ldots\ldots\ldots, a_p \\ b_1, \ldots, b_q, \varphi_0, \ldots \varphi_{2r-1} \end{matrix}\right) dt.$$

Evaluating the *r.h.s.* integral in (3·5) using (3·2) and putting $(2r)^{2r} y = \beta$, we get after some simplification (3·1)

4. In this section we shall evaluate some finite integrals involving G-function. We need a theorem due to Harishanker [6] *viz.*

(4·1) if $\qquad h_1(t) \doteqdot f_1(s)$, $h_2(t) \doteqdot f_2(s)$

and $\qquad t\, h_1(xt)\, h_2(yt) \doteqdot F(s, x, y,)$

then

$$\frac{f_1(s) f_2(s)}{s} = \int_0^{\pi/2} F(s, \cos^2\theta, \sin^2\theta) \sin 2\theta \, d\theta.$$

From [7] we have

$$h_1(t) = e^{-at} t^{\nu-1} G_{p,q}^{l,n}\left(t^r \beta \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right)$$

$$\doteqdot \frac{s(s+a)^{-\nu}}{(2\pi)^{(r-1)/2} r^{-\nu+1/2}} G_{p+r,q}^{l,n+r}\left\{\frac{r^r \beta}{(s+a)^r} \,\middle|\, \begin{matrix} \theta_0, \ldots, \theta_{r-1}\, a_1, \ldots, a_p \\ b_1, \ldots\ldots\ldots\ldots\ldots, b_q \end{matrix}\right\} = f_1(s)$$

where $\quad \theta_R = \dfrac{r-\nu-R}{r}$, $R = 0, 1, \ldots, r-1.$

$R(rb_h+\nu) > 0$, $h = 1, \ldots, l$. $R(a) \geqslant 0$, $p+q < 2(l+n)$,

$$\left|\arg \frac{s^r}{\beta}\right| < \left(l+n+\frac{r}{2}-\frac{p}{2}-\frac{q}{2}\right)\pi \text{ and } r \text{ is a positive integer.}$$

From [3, p. 144] we have

$$h_2(t) = t^{\delta-1} \quad e^{-bt} \doteqdot s\,\Gamma(\delta)(s+b)^{-\delta} = f_2(s)$$

$$R(\delta) < 0,\ R(b) \geqslant 0, \text{ and } R(s) > 0.$$

Applying (4·1) we get

$$(4{\cdot}2) \qquad 2\int_0^{\pi/2} \sin^{2\delta-1}\theta\cos^{2\nu-1}\theta\ (s+a\cos^2\theta+b\sin^2\theta)^{-\nu-\delta}$$

$$\times\, G^{l,n+r}_{p+r,q}\left\{\frac{\beta r^r\cos^{2r}\theta}{(s+a\cos^2\theta+b\sin^2\theta)^r}\middle|\begin{matrix}\varphi_0,\ldots,\varphi_{r-1}, a_1,\ldots,a_p\\ b_1,\ldots\ldots\ldots\ldots\ldots,b_q\end{matrix}\right\}d\theta$$

$$= \Gamma(\delta)r^{-\delta}(s+a)^{-\nu}(s+b)^{-\delta}$$

$$\times\, G^{l,n+r}_{p+r,q}\left\{\frac{r^r\beta}{(s+a)^r}\middle|\begin{matrix}\theta_0,\ldots;\theta_{r-1}, a_1,\ldots,a_p\\ b_1,\ldots\ldots\ldots\ldots\ldots,b_q\end{matrix}\right\}$$

where $\varphi_R = \dfrac{r-\nu-\delta-R}{r}$, $R=0,1,\ldots\ldots,r-1.$

Replacing a by λ's, b by μ's and $\beta\left(\frac{r}{s}\right)^r$ by z and applying a known property [2, p. 109] of G-function we get

$$(4{\cdot}3) \qquad 2\int_0^{\pi/2} \sin^{2\delta-1}\theta\cos^{2\nu-1}\theta\ (1+\lambda\cos^2\theta+\mu\sin^2\theta)^{-\nu-\delta}$$

$$\times\, G^{l,n}_{p,q}\left\{\frac{z\cos^{2r}\theta}{(1+\lambda\cos^2\theta+\mu\sin^2\theta)^r}\middle|\begin{matrix}a_1,\ldots\ldots,a_p\\ b_1,\ldots\ldots,b_q\end{matrix}\right\}d\theta$$

$$= \Gamma(\delta)\,r^{-\delta}(1+\lambda)^{-\nu}(1+\mu)^{-\delta}$$

$$\times\, G^{l,n+r}_{p+r,q+r}\left\{\frac{z}{(1+\lambda)^r}\middle|\begin{matrix}\theta_0,\ldots,\theta_{r-1}, a_1,\ldots,a_p\\ b_1,\ldots,b_q,\varphi_0,\ldots\ldots,\varphi_{r-1}\end{matrix}\right\}$$

provided $R(rb_h+\nu) > 0$, $h=1,2,\ldots\ldots,l$. $R(\delta) > 0$,

$$R(z) > 0,\ \lambda \geqslant 0,\ \mu \geqslant 0,\ p+q < 2(l+n),$$

$|\arg z| < \left(l+n+\frac{r}{2}-\frac{p}{2}-\frac{q}{2}\right)\pi$ and r is a positive integer.

Particular Cases: Put $\cos^2\theta = t$, to get

$$(4\cdot4) \qquad \int_0^1 t^{\nu-1} (1-t)^{\delta-1} \{1+\lambda t+\mu(1-t)\}^{-\nu-\delta} G_{p,q}^{l,n}\left[\frac{zt^r}{\{1+\lambda t+\mu(1-t)\}^r}\left| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right.\right] dt$$

$$= \Gamma(\delta)\, r^{-\delta}(1+\lambda)^{-\nu}(1+\mu)^{-\delta}\, G_{p+r,q+r}^{l,n+r}\left\{\frac{z}{(1+\lambda)^r}\left| \begin{matrix} \theta_0, \ldots, \theta_{r-1}, a_1, \ldots, a_p \\ b_1, \ldots, b_q, \varphi_0, \ldots \varphi_{r-1} \end{matrix}\right.\right\}$$

Put $\lambda=0$, $\mu=1$, and $\sin^2\theta = t$. We get

$$(4\cdot5) \qquad \int_0^1 t^{\delta-1} (1-t)^{\nu-1} (1+t)^{-\nu-\delta} G_{p,q}^{l,n}\left\{z\left(\frac{1-t}{1+t}\right)^r \left| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right.\right\} dt$$

$$= \Gamma(\delta)\, (2r)^{-\delta}\, G_{p+r,q+r}^{l,n+r}\left(z \left| \begin{matrix} \theta_0, \ldots, \theta_{r-1}\; a_1, \ldots, a_p \\ b_1, \ldots, b_q, \varphi_0, \ldots, \varphi_{r-1} \end{matrix}\right.\right)$$

Put $\mu=0$, $\lambda=1$, $\cos^2\theta=t$ and get

$$(4\cdot6) \qquad \int_0^1 t^{\nu-1} (1-t)^{\delta-1}(1+t)^{-\nu-\delta} G_{p,q}^{l,n}\left\{\frac{zt^r}{(1+t)^r}\left| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right.\right\} dt$$

$$= \Gamma(\delta)\, r^{-\delta}\, 2^{-\nu}\, G_{p+r,q+r}^{l,n+r}\left(\frac{z}{2^r}\left| \begin{matrix} \theta_0, \ldots, \theta_{r-1}, a_1, \ldots, a_p \\ b_1, \ldots, b_q, \varphi_0, \ldots, \varphi_{r-1} \end{matrix}\right.\right)$$

Putting $\lambda=\mu=1$ and $\cos^2\theta=t$, $\frac{z}{2^r} = y$ we get

$$(4\cdot7) \qquad \int_0^1 t^{\nu-1} (1-t)^{\delta-1} G_{p,q}^{l,n}\left(t^r y\left| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right.\right) dt$$

$$= \Gamma(\delta)\, r^{-\delta}\, G_{p+r,q+r}^{l,n+r}\left(y \left| \begin{matrix} \theta_0, \ldots, \theta_{r-1}, a_1, \ldots, a_p \\ b_1, \ldots, b_q, \varphi_0, \ldots, \varphi_{r-1} \end{matrix}\right.\right)$$

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ON GENERALIZED LAPLACE TRANSFORMS AND SELF-RECIPROCAL FUNCTIONS

By

R. P. GUPTA

Department of Mathematics, D. S. B. Govt. College, Nainital

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ABSTRACT

The generalization of the Laplace transformation is given by Saxena in the form

$$f(s^{\mu}) = s a\mu \int_0^{\infty} (qs^{\mu} t)^{c-\frac{1}{2}} e^{-(p-\frac{1}{2}q)s^{\mu}t} W_{1,n}(qs^{\mu} t)\, \phi(t)\, dt.$$

If $t^{\lambda} \phi(t)$ is self-reciprocal in $\chi_{\nu,k,m}$ - transform then

$$f(s^{\mu}) = \int_0^{\infty} y^{\lambda+\frac{1}{2}} \left\{ g_{\lambda}^{\mu}(y/s^{\mu}) + G_{\lambda}^{\mu}(y/s^{\mu}) \phi(y)\, dy.\right.$$

1. INTRODUCTION

A generalization of the Laplace transform

$$\phi(s) = s \int_0^{\infty} e^{-st} f(t)\, dt \tag{1·1}$$

has been introduced by Saksena [1] in the form

$$\phi(s^{\mu}) = s a\mu \int_0^{\infty} (qs^{\mu}t)^{c-\frac{1}{2}} e^{-(p-\frac{1}{2}q)s^{\mu}t} W_{l,n}(qs^{\mu}t) f(t)\, dt, \tag{1·2}$$

where $W_{l,n}(x)$ is a Whittaker function.

For suitable values to the parameters, (1·2) reduces to (1·1)

The object of this paper is to obtain the generalized Laplace transform of $f(x)$, when $x^{\lambda} f(x)$ is self-reciprocal in $\chi_{\nu,k,m}$ - transform due to Roop Narain [2], *viz.*,

$$f(x) = \frac{1}{2^{\nu}} \int_0^{\infty} (xy)^{\nu+\frac{1}{2}} \chi_{\nu,k,m}(\tfrac{1}{4} x^2 y^2)\, \phi(y)\, dy \tag{1·3}$$

where

$$\chi_{\nu,k,m}(x) = \frac{\Gamma(2m)\,\Gamma(\frac{3}{2}+\nu+m-k)}{\Gamma(\frac{1}{2}-k+m)\,\Gamma_*(\nu+1+m\pm m)}\, {}_2F_3\left[\begin{matrix}\frac{1}{2}+k-m, \frac{3}{2}+\nu+m-k\,; \\ 1-2m, \nu+1+m\pm m\,;\end{matrix} -x\right]$$

$$+ \frac{\Gamma(-2m)\,\Gamma(\frac{3}{2}+\nu+3m-k)}{\Gamma(\frac{1}{2}-k-m)\,\Gamma_*(\nu+1+3m\pm m)}\, {}_2F_3\left[\begin{matrix}\frac{1}{2}+k+m, \frac{3}{2}+\nu+3m-k\,; \\ 1+2m, \nu+1+3m\pm m\,;\end{matrix} -x\right] \tag{1·4}$$

$$= x^{-\nu}\, G^{2,1}_{2,4}\left(x \,\middle|\, \begin{matrix} k-m-\frac{1}{2}, \frac{1}{2}+\nu-k+m \\ \nu, \nu+2m, -2m, 0 \end{matrix}\right) \tag{1·5}$$

For the sake of brevity the symbol $\Gamma_*(a\pm b)$ is used to denote $\Gamma(a+b)\,\Gamma(c-b)$.

For $k + m = \frac{1}{2}$ an $R_\nu(k, m)$ function reduces to an R_ν function.

2. *Theorem.* If

$$f(s^\mu) = s^{a\mu} \int_0^\infty (q s^\mu t)^{c - \frac{1}{2}} e^{-(p - \frac{1}{2} q) s^\mu t} W_{l,n}(q s^\mu t)\, \phi(t)\, dt \tag{2·1}$$

and $t^\lambda \phi(t)$ is self-reciprocal in $\chi_{\nu,k,m}$ – transform,

then

$$f(s^\mu) = \int_0^\infty y^{\lambda + \frac{1}{2}} \left\{ g_\lambda^\mu (y/s^\mu) + G_\lambda^\mu (y/s^\mu) \right\} \phi(y)\, dy \tag{2·2}$$

where

$$g_\lambda^\mu (y/s^\mu) = \frac{\Gamma(2m)\, \Gamma(1 - 2m)\, s^{\mu(a + \lambda - \frac{3}{2})}\, q^{\lambda - \nu - \frac{3}{2}}}{2^{-\nu}\, \Gamma(\frac{1}{2} - k + m)\, \Gamma(\frac{1}{2} + k - m)} \left(\frac{y}{4s^\mu}\right)^\nu \times$$
$$\times \sum_{r=0}^\infty \frac{(-1)^r}{r!} \frac{\Gamma(\frac{1}{2} + k - m + r)\, \Gamma(\frac{3}{2} + \nu + m - k + r)\, \Gamma_*(c \pm n + \nu - \lambda + 2r + \frac{3}{2})}{2^{-2\nu}\, \Gamma(1 - 2m + r)\, \Gamma_*(1 + \nu + m \pm m + r)\, \Gamma(c + \nu - \lambda + 2r - l + 2)\, q^{2r}}$$
$$\times \left(\frac{y}{4s^\mu}\right)^{2r} {}_2F_1 \left[\begin{matrix} c \pm n + \nu - \lambda + 2r + \frac{3}{2}\, ; \\ c + \nu - \lambda + 2r - l + 2\, ; \end{matrix} \left(1 - \frac{p}{q}\right) \right] \tag{2·3}$$

and

$$G_\lambda^\mu (y/s^\mu) = \frac{\Gamma(-2m)\, \Gamma(1 + 2m)\, s^{\mu(a + \lambda - \frac{3}{2})}\, q^{\lambda - \nu - 4m - \frac{3}{2}}}{\Gamma(\frac{1}{2} - k - m)\, \Gamma(\frac{1}{2} + k + m)\, 2^{-\nu - 4m}} \left(\frac{y}{4s^\mu}\right)^{\nu + 4m} \times$$
$$\times \sum_{r=0}^\infty \frac{(-1)^r}{r!} \frac{\Gamma(\frac{1}{2} + k + m + r)\, \Gamma(\frac{3}{2} + \nu + 3m - k + r)\, \Gamma_*(c \pm n + \nu - \lambda + 2m + 2r + \frac{3}{2})}{q^{2r}\, 2^{-2r}\, \Gamma(1 + 2m + r)\, \Gamma_*(1 + \nu + 3m \pm m)\, \Gamma(c + \nu - \lambda + 2m + 2r - l + 2)}$$
$$\times \left(\frac{y}{4s^\mu}\right)^{2r} {}_2F_1 \left[\begin{matrix} c \pm n + \nu - \lambda + 2m + 2r + \frac{3}{2}\, ; \\ c + \nu - \lambda + 2m + 2r - l + 2\, ; \end{matrix} \left(1 - \frac{p}{q}\right) \right] \tag{2·4}$$

provided

$$\left. \begin{aligned} \phi(x) &= 0(x^\alpha) \text{ for small } x \\ &= 0(e^{-bx}) \text{ for large } x \end{aligned} \right\} ; Re\, b > 0$$

$Re(\lambda + \nu + \alpha \pm 2m + \frac{3}{2}) > 0$, $Re(c + \nu - \lambda \pm n \pm 2m + \frac{3}{2}) > 0$

$Re(\nu - \lambda + c \pm n + \frac{3}{2}) > 0$, $Re(\nu + c - \lambda \pm n + 4m + \frac{3}{2}) > 0$

$Re(c \pm n + \alpha + 1) > 0$, $Re(p s^\mu) > 0$; $p \geqslant q$.

Proof : Since

$$f(s^\mu) = s^{a\mu} \int_0^\infty (q s^\mu t)^{c - \frac{1}{2}} e^{-(p - \frac{1}{2} q) s^\mu t} W_{l,n}(q s^\mu t)\, \phi(t)\, dt$$

and

$$t^\lambda \phi(t) = \frac{1}{2^\nu} \int_0^\infty (t y)^{\nu + \frac{1}{2}} \chi_{\nu, k, m}(\tfrac{1}{4} t^2 y^2)\, y^\lambda \phi(y)\, dy,$$

we have

$$f(s^{\mu}) = 2^{-\nu} s^{a\mu} \int_0^{\infty} (q\, s^{\mu}\, t)^{c-\frac{1}{2}}\, e^{-(p-\frac{1}{2}q)\, s^{\mu} t}\, W_{l,n}(q\, s^{\mu}\, t)\, dt$$

$$\times \int_0^{\infty} t^{-\lambda}\, (ty)^{\nu+\frac{1}{2}}\, \chi_{\nu,k,m}(\tfrac{1}{4} t^2 y^2)\, y^{\lambda}\, \phi(y)\, dy, \tag{2·5}$$

$$= 2^{-\nu}\, s^{\mu(a+c-\frac{1}{2})}\, q^{c-\frac{1}{2}} \int_0^{\infty} y^{\lambda+\nu+\frac{1}{2}}\, \phi(y)\, dy$$

$$\times \int_0^{\infty} t^{c-\lambda+\nu}\, e^{-(p-\frac{1}{2}q)\, s^{\mu}\, t}\, W_{l,n}(q\, s^{\mu}\, t)\, \chi_{\nu,k,m}(\tfrac{1}{4} t^2 y^2)\, dt, \tag{2·6}$$

provided the change in the order of integration is justified.

But t–integral

$$= \int_0^{\infty} t^{c-\lambda+\nu}\, e^{-(p-\frac{1}{2}q)\, s^{\mu}\, t}\, W_{l,n}(q\, s^{\mu}\, t)\, \chi_{\nu,k,m}(\tfrac{1}{4} t^2 y^2)\, dt,$$

$$= \frac{\Gamma(2m)\, \Gamma(\frac{3}{2}+\nu+m-k)}{\Gamma(\frac{1}{2}-k+m)\, \Gamma_*(\nu+1+m\pm m)}\, I_1 + \frac{\Gamma(-2m)\, \Gamma(\frac{3}{2}+\nu+3m-k)}{\Gamma(\frac{1}{2}-k+m)\, \Gamma_*(1+\nu+3m\pm m)}\, I_2 \text{ (say)},$$

where

$$I_1 = \int_0^{\infty} t^{c-\lambda+\nu} e^{-(p-\frac{1}{2}q)\, s^{\mu}\, t}\, W_{l,n}(q\, s^{\mu}\, t)\, {}_2F_3\left[\begin{matrix}\frac{1}{2}+k-m,\ \frac{3}{2}+\nu+m-k\,; \\ 1-2m,\ 1+\nu+m\pm m\,;\end{matrix} -(\tfrac{1}{4} t^2 y^2)\right] dt$$

$$+ \sum_{r=0}^{\infty} \frac{(-1)^r}{r!}\, \frac{\Gamma(\frac{1}{2}+k-m+r)\, \Gamma(\frac{3}{2}+\nu+m-k+r)\, \Gamma(1-2m)\, \Gamma_*(\nu+1+m\pm m)}{\Gamma(\frac{1}{2}+k-m)\, \Gamma(\frac{3}{2}+\nu+m-k)\, \Gamma(1-2m+r)\, \Gamma_*(\nu+1+m\pm m+r)}\, (\tfrac{1}{4} y^2)^r$$

$$\times \int_0^{\infty} t^{\nu-\lambda+c+2r}\, e^{-(p-\frac{1}{2}q)\, s^{\mu}\, t}\, W_{l,n}(q\, s^{\mu}\, t)\, dt \tag{2·7}$$

and

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!}\, \frac{\Gamma(\frac{1}{2}+k-m+r)\, \Gamma(\frac{3}{2}+\nu+m-k+r)\, \Gamma(1-2m)\, \Gamma_*(1+\nu+m\pm m)}{\Gamma(\frac{1}{2}+k-m)\, \Gamma(\frac{3}{2}+\nu+m-k)\, \Gamma(1-2m+r)\, \Gamma_*(1+\nu+m\pm m+r)} \left(\frac{y^2}{4}\right)^r$$

$$\times \frac{\Gamma_*(c\pm n+\nu-\lambda+2r+\frac{3}{2})\; q^{-c+\lambda-\nu-2r-1}}{\Gamma(c+\nu-\lambda+2r-l+2)\; S^{\mu(c-\lambda+\nu+2\gamma+1)}}$$

$$\times\, {}_2F_1\left[\begin{matrix} c\pm n+\nu+2r-\lambda+\frac{3}{2}\,; \\ c+\nu-\lambda+2r-l+2\,; \end{matrix} \left(1-\frac{p}{q}\right)\right] \tag{2·8}$$

on making use of the Goldsteins integral [3], provided

$$Re(c \pm n + \nu - \lambda + \tfrac{3}{2}) > 0.$$

And

$$I_2=4^{-2m}\ y^{4m}\int_0^\infty t^{c-\lambda+\nu+4m}\ e^{-(p-\frac{1}{2}q)\,s^{\mu}t}\ W_{l,n}(q\,s^{\mu}\,t)$$

$$\times\ {}_2F_3\left[\begin{matrix}\frac{1}{2}+k+m,\ \frac{3}{2}+\nu+3m-k\ ; & -\dfrac{t^2y^2}{4}\\ 1+2m,\ 1+\nu+3m\pm m\ ; & \end{matrix}\right]dt$$

$$=\frac{\Gamma(1+2m)\ \Gamma_*(1+\nu+3m\pm m)\ y^{4m}}{\Gamma(\frac{1}{2}+k+m)\ \Gamma(\frac{3}{2}+\nu+3m-k)4^{2m}}$$

$$\times\sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!}\ \frac{\Gamma(\frac{1}{2}+k+m+r)\ \Gamma(\frac{3}{2}+\nu+3m-k+r)\ y^{2r}}{4^r\ \Gamma(1+2m+r)\ \Gamma_*(1+\nu+3m\pm m+r)}$$

$$\times\ \frac{\Gamma_*(c\pm n+\nu-\lambda+4m+2r+\frac{3}{2})\ q^{-c+\lambda-\nu-2r-4m-1}}{\Gamma(c-\lambda+\nu+4m+2r-l+2)\ s^{\mu(c-\lambda+\nu+4m+2r+1)}}$$

$$\times\ {}_2F_1\left[\begin{matrix}c\pm n+\nu-\lambda+4m+2r+\frac{3}{2}\ ; & \left(1-\dfrac{p}{q}\right)\\ c+\nu-\lambda+4m+2r-l+2\ ; & \end{matrix}\right] \tag{2·9}$$

provided $Re\ (c\pm n+\nu-\lambda+4m+\frac{3}{2})>0$.

Hence the result.

The change in the order of integration in (2·3) can be justified as follows :

Let $\theta(y)=y^{\lambda+\nu+\frac{1}{2}}\ \phi(y)\int_0^\infty t^{\nu+c-\lambda}\ e^{-(p-\frac{1}{2}q)s^{\mu}t}\ W_{l,n}(q\,s^{\mu}\,t)\,\chi_{\nu,k,m}(\frac{1}{4}y^2t^2)\ dt$

$\psi(t)=t^{\nu+c-\lambda}\ e^{-(p-\frac{1}{2}q)\,s^{\mu}\,t}\ W_{l,n}(qs^{\mu}t)\int_0^\infty y^{\lambda+\nu+1/2}\,\phi(y)\chi_{\nu,k,m}(\frac{1}{4}\,t^2\,y^2)\ dt$

and

$$\left.\begin{aligned}\phi(x)&=0\,(x^{\alpha})\ \text{for small } x\\ &=0\,(e^{-bx})\text{for large } x\end{aligned}\right\}\ Re\ b>o$$

Now $\theta(y)$ converges uniformly for $y\geqslant 0$, if $Re\ (\lambda+\nu+\alpha+\frac{1}{2}\pm 2m)>0$,

$Re\ (\nu+c-\lambda\pm 2n\pm 2m+\frac{3}{2})>0$, $Re\ (s^{\mu}p)>0$, $p\geqslant q$, on account of the following behaviour of $W_{k,m}(x)$ and $\chi_{\nu,k,m}(x)$

$$W_{k,m}(x)=0\,(x^{\pm m+\frac{1}{2}})\ \text{for small } x$$

$$=0\,(x^{k}\ e^{-\frac{1}{2}x})\ \text{for large } x$$

and

$$\chi_{\nu,k,m} = 0\,(x^{m \pm m}) \text{ for large } x$$

$$= 0\,(x^{-\xi}) \text{ for large } x$$

where $\xi = \frac{1}{2}(\frac{1}{2}+\nu)$, $(\frac{1}{2}+k-m)$ or $(\frac{3}{2}+\nu+m-k)$;

and $\psi(t)$ converges uniformly for $t \geqslant 0$, if $Re(\nu+c-\lambda \pm n+\frac{1}{2}) > 0$,

$Re(\lambda+\nu+\alpha \pm 2m+\frac{3}{2}) > 0$, $Re(s^{\mu} p) > 0, p \geqslant q$.

Again if we consider the integral

$$I = \frac{S^{\mu(a+c-\frac{1}{2})}}{2^{\nu}} q^{c-\frac{1}{2}} \int_0^{\infty} | y^{\lambda+\nu+\frac{1}{2}} \phi(y) | \, dy$$

$$\times \int_{T'}^{\infty} | t^{c+\nu-\lambda} e^{-(p-\frac{1}{2}q)s^{\mu}t} W_{l,n}(q s^{\mu} t) \chi_{\nu,k,m}(\tfrac{1}{4}t^2 y^2) | \, dt$$

where T and T' are large, we see, on account of asymptotic behaviour of $W_{l,n}(x)$ and $\chi_{\nu,k,m}(x)$, that I does not exceed a constant multiple of

$$\frac{S^{\mu(a+c-\frac{1}{2})}}{2^{\nu}} q^{c-\frac{1}{2}} \int_0^{\infty} | y^{\lambda+\nu+\frac{1}{2}} \phi(y) | \, dy \int_T^{\infty} | t^{c+\nu-\lambda+l-\xi} e^{-ps^{\mu}t} | \, dt$$

which tends to zero, when $Re(s^{\mu} p) > 0$.

The change in the order of integration and summation in (2·7) can be justified as follows :

(*i*) The hypergeometric function ${}_2F_3$ is an integral function and consequently uniformly convergent in an arbitrary interval $(0, \varepsilon)$

(*ii*) $t^{c-\lambda+\nu} e^{-(p-\frac{1}{2}q)s^{\mu}t} W_{l,n}(qs^{\mu}t)$ is continuous for all finite $t \geqslant 0$,

provided $Re(c-\lambda+\nu \pm n+\frac{1}{2}) > 0$;

(*iii*) $2^{-\nu} s^{a\mu} \int_0^{\infty} (qs^{\mu}t)^{c-\frac{1}{2}} t^{-\lambda+\nu+\frac{1}{2}} e^{-(p-\frac{1}{2}q)S^{\mu}t} W_{l,n}(qs^{\mu}t) \chi_{\nu,k,m}(\frac{1}{4}t^2y^2) \, dt$

is also convergent, when $Re(c-\lambda \pm \nu+n \pm 2m+\frac{3}{2}) > 0$.

(*iv*) The integral in question converges on account of the asymptotic behaviour of the function

$$W_{k,m} \sim x^{k} e^{-\frac{1}{2}x}$$

This completes the proof of our theorem.

3. Properties of $g^{\mu}_{\lambda}(t)$ and $G^{\mu}_{\lambda}(t)$, where $t = (y/s^{\mu})$.

The behaviour of $g^{\mu}_{\lambda}(t)$ for small and large values of t can easily be investigated. We have

$$g^{\mu}_{\lambda}(t) = \frac{\Gamma(2m)\ \Gamma(1-2m)\ s^{\mu(a+\lambda-3/2)}\ t^{\nu}}{2^{\nu}\ \Gamma(\frac{1}{2}-k+m)\ \Gamma(\frac{1}{2}+k-m)\ q^{\nu-\lambda+3/2}}$$

$$\times \sum_{r=0}^{\infty} \frac{(-1)^r\ \Gamma(\frac{1}{2}+k-m+r)\ \Gamma(\frac{3}{2}+\nu+m-k+r)\ \Gamma_*(c\pm n+\nu-\lambda+2r+\frac{3}{2})}{r!\ (2q)^{2r}\ \Gamma(1-2m+r)\ \Gamma_*(\nu+1+m\pm m+r)\ \Gamma(c+\nu-\lambda+2r-l+2)}$$

$$\times\ t^{2r}\ {}_2F_1\left[\begin{matrix} c\pm n+\nu-\lambda+2r+\frac{3}{2}\ ; \\ c+\nu-\lambda+2r-l+2\ ; \end{matrix}\left(1-\frac{p}{q}\right)\right]$$

$$= \frac{\Gamma(2m)\ \Gamma(1-2m)\ s^{\mu(a+\lambda-3/2)}\ t^{\nu}}{2^{\nu}\ \Gamma(\frac{1}{2}-k+m)\ \Gamma(\frac{1}{2}+k-m)\ q^{\nu-\lambda+3/2}}$$

$$\times \sum_{0}^{\infty} \frac{(-1)^r\ \Gamma(\frac{1}{2}+k-m+r)\ \Gamma(\frac{3}{2}+\nu+m-k+r)\ \Gamma(c+n+\nu-\lambda+2r+\frac{3}{2})}{r!\ 2^{2r}\ \Gamma(1-2m+r)\ \Gamma_*(\nu+1+m\pm m+r)}$$

$$\times \frac{t^{2r}}{\left(\frac{p}{q}\right)^{l+c+\nu-\lambda+1} p^{2r}\ \Gamma(n-l+\frac{1}{2})} \int_0^1 x^{n-l-\frac{1}{2}}\ (1-x)^{c+\nu-\lambda-n+2r+\frac{1}{2}}$$

$$\times \left[1-\left(1-\frac{p}{q}\right)x\right]^{l+n-\frac{1}{2}} dx$$

by virtue of the relations

$${}_2F_1(\alpha, \beta\ ; \gamma\ ; x) = (1-x)^{\gamma-\alpha-\beta}\ {}_2F_1(\gamma-\alpha, \gamma-\beta\ ; \gamma\ ; x)$$

and

$${}_2F_1(a, b\ ; c\ ; z) = \frac{\Gamma(c)}{\Gamma(c-a)\ \Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\ (1-zt)^{-a}\, dt$$

Hence

$$g^{\mu}_{\lambda}(t) = K_1\, t^{\nu} \int_0^{\infty} x^{n-l-\frac{1}{2}}\ (1-x)^{c+\nu-\lambda-n+\frac{1}{2}} \left\{1-\left(1-\frac{p}{q}\right)x\right\}^{l+n-\frac{1}{2}}$$

$$\times\ {}_4F_3\left[\begin{matrix}\frac{1}{2}+k-m,\ \frac{3}{2}+\nu+m-k,\ \frac{1}{2}c+\frac{1}{2}n+\frac{1}{2}\nu-\frac{1}{2}\lambda+\frac{3}{4},\ \frac{1}{2}(c+n+\nu-\lambda)+\frac{5}{4}\ ; \\ 1-2m,\ 1-\nu,\ 1+\nu+2m\ ;\ -(t^2/4p^2)\,(1-x)^2\end{matrix}\right] dx$$

where K_1 is constant independent of x.

It follows by the definition of $g^{\mu}_{\lambda}(t)$ that

$g^{\mu}_{\lambda}(t) = 0\,(t)^{\nu}$ for small values of t and from the asymptotic expansion of ${}_4F_3$ we have

$$g^{\mu}_{\lambda}(t) = 0\ (t^{-2\delta+\nu}) \text{ for large value of } t$$

where $\delta = \min\,(\frac{1}{2}+k-m,\ \frac{3}{2}+\nu+m-k,\ \frac{1}{2}c+\frac{1}{2}n+\frac{1}{2}\nu-\frac{1}{2}\lambda+\frac{3}{4},\ \frac{1}{2}c+\frac{1}{2}n+\frac{1}{2}\nu-\frac{1}{2}\lambda+\frac{5}{4})$.

The behaviour of $G^{\mu}_{\lambda}(t)$ can be investigated in the same manner.

4. *Particular Cases*

(*i*) For $k+m=\frac{1}{2}$, the $\chi_{\nu,k,m}$ – transform reduces the Hankel transform and all the theorems and corallaries due to Snehlata [4] can be deduced as the particular cases of the result obtained in § 2.

(*ii*) For suitable values to the parameters, (1·2) reduces to the generalized Laplace transform given by R. S. Varma [5] and the theorems and corallaries due to S. Masood [6] can also be deduced as particular cases of the result obtained in § 2.

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ON PSEUDO-STATIONARY GAS FLOWS

By

A. INDRASENA

Department of Mathematics, Osmania University, Hyderabad-7 A. P.

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ABSTRACT

Considering the geometric theory of triply orthogonal spatial curves, one of them related to the streakline, the kinematical properties of thermally nonconducting, inviscid, with two specific heats constant, unsteady flows in the absence of extraneous forces studied are :

(1) Transforming the basic equations governing the flow, defined above, in the intrinsic form we observe that the pressure is uniform along the binormal to the streakline.

(2) Expressions for vorticity components along the principal normal and binormal to the streakline are obtained in terms of velocity and curvature of the streakline.

(3) Velocity along the streakline decreases in the subsonic region when the normal congruences are minimal.

(4) Studying the Beltrami velocity vector field related to the streaklines we have proved the streakline patterns to be either right circular helices, or circles or parallel lines. Also velocity, curvature and torsion of the streakline are constant in the osculating plane. Pressure and density decrease along the streakline.

(5) Geometrical conditions are obtained for a doubly laminar vector field.

(6) Intrinsic properties of plane flows are studied and elegant approaches are indicated.

(7) Finally, the compatability conditions to be satisfied by pseudo-stationary gas flows have been derived.

§ 1. INTRODUCTION

The kinematical properties of thermally non-conducting stationary gas flows have been studied extensively by various authors [2,3,6–11] in the field of fluid mechanics. Truesdell [11] surveying the literature existing, has outlined simple approaches to the problems of gasdynamics. Prem Kumar [6,7] has studied the intrinsic equations of relativistic gas. Purushotham [8] has furnished the kinematical conditions existing for viscous, incompressible steady flows. Prem Kumar [7] following Taub [10] has considered some of the intrinsic properties of pseudo-stationary gas flows. But the detailed investigations for plane as well as spatial flows yet to be considered. Consequently, herein we have investigated some of the kinematical properties of pseudo stationary gas flows. The basic equations governing the flow are transformed into intrinsic form, Beltrami velocity vector field is considered and streakline patterns are obtained. Doubly laminar vector field is studied. Properties of plane flows are investigated. Finally, the compatibility conditions to be satisfied by pseudo-stationary gas flows, analogous to Berker [1], have been derived.

§ 2 (A) BASIC EQUATIONS

The basic equations governing the motion of unsteady, thermally non-conducting, inviscid gas flows with two specific heats constant are given below, in the absence of external forces, in the usual notation [5] : as

(1) $$\frac{d\rho}{dt} + \rho \operatorname{div} \vec{q} = 0$$

(2) $$\frac{d\vec{q}}{dt} + \frac{1}{\rho} \nabla p = 0$$

(3) $$\frac{dS}{dt} = 0$$

(4) $$\rho = \rho(p, S)$$

Where $\vec{q}$ denotes the velocity vector, ρ the fluid density, p the pressure t the time and S the entropy.

(B) TRANSFORMATION

Let us define the position vectors $\vec{r}\,(x, y\ z)$ and $\vec{R}\ (\alpha_1, \beta_1, \gamma_1)$, (where $\alpha_1 = x/t$, $\beta_1 = y/t$, $\gamma_1 = z/t$) in triply orthogonal coordinate system. The following transformations are useful to transform the above equations into current coordinate system $(\alpha_1, \beta_1, \gamma_1)$

(5) $$\nabla = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) = \frac{1}{t}\left(i\frac{\partial}{\partial \alpha_1} + j\frac{\partial}{\partial \beta_1} + k\frac{\partial}{\partial \gamma_1} \right) = \frac{\overset{/}{\nabla}}{t}$$

(6) $$\frac{\partial}{\partial t} = -\frac{1}{t}(\vec{R}.\overset{/}{\nabla})$$

Using (5) and (6) we have

(7) $$\frac{d}{dt} = \frac{\vec{Q}}{t}.\overset{/}{\nabla}$$

The velocity vector is given by

(8) $$\frac{d\vec{r}}{dt} = \vec{q} = \vec{Q} + \vec{R}$$

Using (5) – (8) the above equations (1) – (3) can be transformed as

(9) $$\vec{Q}.\overset{/}{\nabla} \log \rho + 3 + \overset{/}{\nabla}.\vec{Q} = 0$$

(10) $$\frac{1}{\rho} \overset{/}{\nabla} p + \vec{Q} + (\vec{Q}.\overset{/}{\nabla})\vec{Q} = 0$$

(11) $$\left(\vec{Q}.\overset{/}{\nabla}\right) S = 0$$

§ 3. INTRINSIC TRANSFORMATION

(*A*) Geometrical and Vectoral relations : Considering $\vec{s}$, $\vec{n}$ and $\vec{b}$ as triply orthogonal unit tangent vectors along the curves of congruences formed by the streaklines and their principal normals and binormals respectively and denoting $\frac{d}{ds}$, $\frac{d}{dn}$, $\frac{d}{db}$ as directional derivatives along these vectors, also selecting $\vec{R}$ as the

position vector in space we have the following relations analogous to the geometry of the streamlines[8] :

(12) $$\frac{d\vec{R}}{ds} = \frac{\vec{Q}}{Q} = \vec{s}$$

(13) $$\nabla' . \vec{s} = -(K' + K'')$$

(14) $$\nabla' . \vec{n} = -K$$

(15) $$\nabla' \vec{b} = 0$$

(16) $$\text{curl } \vec{s} = \vec{s}\,(\sigma' - \sigma'') + \vec{b}\,K$$

(17) $$\text{curl } \vec{n} = -\vec{n}\,(\sigma'' + \tau) - \vec{b}\,K'$$

(18) $$\text{curl } \vec{b} = \vec{n}\,K'' + \vec{b}\,(\sigma' - \tau)$$

Using the solenoidal property of (16) – (18) we have

(19) $$\nabla'.\left(\text{curl } \vec{s}\right) = \frac{dK}{db} + \frac{d}{ds}(\sigma' - \sigma'') + (K' + K'')\,\sigma'' - \sigma') = 0$$

(20) $$\nabla'.\left(\text{curl } \vec{n}\right) = K(\sigma'' + \tau) - \frac{d}{dn}(\sigma'' + \tau) - \frac{dK'}{db} = 0$$

(21) $$\nabla'\left(\text{curl } \vec{b}\right) = \frac{dK''}{dn} - K\,K'' + \frac{d}{db}(\sigma' - \tau) = 0$$

where (K, K', K'') and $(\tau, \sigma' \sigma'')$ are curvatures and torsions of the above curves.

Also

(22) $$\text{curl } \vec{Q} = \vec{s}\,Q(\sigma' - \sigma'') + \vec{n}\,\frac{dQ}{db} + \vec{b}\left(KQ - \frac{dQ}{dn}\right)$$

(22) $$= \vec{s}\,\zeta s + \vec{n}\,\zeta n + \vec{b}\,\zeta b$$

where ζ_s, ζ_n and ζ_b are the resolved parts of vorticity components along the above vectors.

(*B*) Using the above relations we have the following intrinisic relations :

(23) $$Q\frac{d\rho}{ds} + \rho\frac{dQ}{ds} + 3\rho = Q\rho\,(K' + K'')$$

(24) $$\frac{1}{\rho}\frac{dp}{ds} + Q\left(1 + \frac{dQ}{ds}\right) = 0$$

(25) $$\frac{1}{\rho}\frac{dp}{dn} + KQ^2 = 0$$

(26) $$\frac{dp}{db} = 0$$

(27) $$\frac{dS'}{ds} = 0$$

This approach is more elegant than the one considered by Prem Kumar[7]

From (24) we observe that the uniformity of velocity along the streakline does not imply the uniformity of the pressure and the converse. When the streaklines are straight, from (25) and (26) we observe that the pressure varies only along the streaklines. The relation (26) shows that the pressure is uniform in the osculating plane as in the case of stationary flows.

Taking the scalar product of (10) by $\vec{n}$ and $\vec{b}$, and using (25) we obtain

(28) $$Q\zeta b = KQ^2 - \tfrac{1}{2}\frac{dQ^2}{dn}$$

(29) $$Q\zeta n = \tfrac{1}{2}\frac{dQ^2}{db}$$

Assuming, in particular, the form of the equation of state (4 to be $p = \rho\, e, \gamma S/Jcv$ and operating $\overset{/}{\nabla}$ multiplying the resultant equation by $\vec{s}$, $\vec{n}$, $\vec{b}$ scalarly and using (26) and (27) we obtain

(30) $$\frac{dp}{ds} = \frac{\gamma p}{\rho}\frac{d\rho}{ds}$$

(31) $$\frac{1}{p}\frac{dp}{dn} = \frac{\gamma}{\rho}\frac{d\rho}{dn} + \frac{1}{Jcv}\frac{dS}{dn}$$

(32) $$0 = \frac{\gamma}{\rho}\frac{d\rho}{db} + \frac{1}{Jcv}\frac{dS}{db}$$

where γ is ratio of the specific heats, J_{cv} the Joules' Constant and cv the specific heat at constant volume. From (30) we observe that the uniformity of pressure along the streakline implies the uniformity of density in the same direction.

Using (23), (24) and (30) we have

(33) $$\frac{dp}{ds} = \frac{Q\rho}{M^2}\left\{Q\,(K' + K'') - 3 - \frac{dQ}{ds}\right\} = -\rho Q\left(1 + \frac{dQ}{ds}\right)$$

where $M = Q/c$ in the Mach number.

Using the last two equations of (33) we obtain

(34) $$\frac{dQ}{ds} = \frac{M^2 + Q(K'+K'') - 3}{(1 - M^2)}$$

If the normal surfaces are minimal *i.e.* $(K'+K'') = 0$. We obtain from (34)

(35) $$\frac{dQ}{ds} = \frac{M^2 - 3}{1 - M^2}$$

From the above relation we infer that the velocity of the streakline decreases along the streakline in the subsonic region when the normal surfaces are minimal.

Eliminating $\frac{dp}{ds}$ from (24) and (30) we obtain

$$\frac{d\rho}{ds} = -\frac{\rho Q}{c^2}\left(1 + \frac{dQ}{ds}\right) = -\frac{\rho M^2}{Q}\left(1 + \frac{dQ}{ds}\right) \tag{36}$$

Eliminating $\frac{dp}{dn}$ from (25) and (31) we have

$$\frac{d\rho}{dn} = -\rho\left(KM^2 + \frac{1}{Jcv}\frac{dS}{dn}\right) \tag{37}$$

Equations (36) and (37) and (32) give the vatiation of density along the streakline, principal normal and binormal respectively.

(*B*) *Beltrami vector field.*—The vector $\vec{Q}$ is defined to be a Beltrami vector field[4] if

$$\text{curl}\,\vec{Q} = \lambda\vec{Q} \tag{38}$$

In this case we obtain

$$\lambda = \sigma' - \sigma'' \tag{39}$$

$$\frac{dQ}{db} = 0 \tag{40}$$

$$KQ - \frac{dQ}{dn} = 0 \tag{41}$$

where 'λ' need not be a constant

From (40) and (41) we obtain

$$\frac{dK}{db} = 0 \tag{42}$$

Using (40), (25) and (26) we have

$$\frac{d\rho}{ds} = 0 \tag{43}$$

From (42) and (32) we obtain

$$\frac{dS}{db} = 0 \tag{44}$$

Using the solenoidal property of (38) together with (19) we have

$$\frac{dQ}{ds} = 0 \tag{45}$$

Using (45) in (41) we obtain

$$\frac{dK}{ds} = 0 \tag{46}$$

Following Weatherburn[12] we have

$$a(\tau^2 + K^2) = K \tag{47}$$

From (46) and (47) we have

$$\frac{d\tau}{ds} = 0 \tag{48}$$

From (46) and (48) we conclude that the streaklines are either right circular helices or a system of concentric circles or parallel lines.

Using (45) in (24) we get

$$\frac{dp}{ds} = - \rho Q \tag{49}$$

From (49) and (25) we infer that the pressure decreases in the osculating plane.

Using (45) in (36) we have

$$\frac{d\rho}{ds} = - \frac{\rho M^2}{Q} \tag{50}$$

This shows that the density drcreases along streakline.

(*C*) *Doubly laminar field.*—A doubly laminar field is defined to be[8]

$$\vec{Q} = \alpha \overset{\prime}{\nabla} \Phi \tag{51}$$

where α and Φ are scalar point functions.

For this field the condition for the streaklines to be intersected normally by a one parameter family of surfaces, *viz.*

$$\vec{Q} \operatorname{curl} \vec{Q} = \overset{\prime}{\nabla} \Phi \operatorname{curl} \vec{Q} = 0 \tag{52}$$

This shows that the Beltrami surface contains vortex lines. This has been observed by Suryanarayan[9] for gas flows. Also (52) yields

$$\sigma' - \sigma'' = 0 \tag{53}$$

Using this in (19) we obtain

$$\frac{dK}{db} = 0 \tag{54}$$

Relations (53) and (54) are the kinematical relations to be satisfied by a doubly laminar field.

§ 4. PLANE FLOWS

We shall consider two plane orthogonal curves for which also denoting $\frac{d}{ds}$ and $\frac{d}{dn}$ as directional derivatives along these curves in $\alpha_1 \partial \beta_1-$ plane.

(A) *Vorticity expression.*—Let u and γ be the velocity components along α_1 and β_1 directions.

From geometrical considerations the operator gradient can be expressed as

$$i\frac{\partial}{\partial\alpha_1}+j\frac{\partial}{\partial\beta_1}=(i\cos\omega+j\sin\omega)\frac{d}{ds}+(-i\sin\omega+j\cos\omega)\frac{d}{dn} \tag{55}$$

where ω is the angle between the direction of flow and α_1 axis. We shall define

$$\zeta = -\operatorname{curl}\vec{Q}\ . \tag{56}$$

where ζ is the magnitude of the vorticity of the pseudo-stationary velocity vector and is perpendicular to the plane of flow.

Now using (55) and (56) we obtain

$$\zeta=-Q\frac{d\omega}{ds}+\frac{dQ}{ds}=\frac{dQ}{dn}-KQ \tag{57}$$

Using (9) in (55) we get

$$Q\frac{d\rho}{ds}+\rho\frac{dQ}{ds}+2\rho=-Q\rho\frac{d\omega}{dn} \tag{58}$$

which is the equation of continuity for plane pseudo-stationary flow.

Using (10) in (55) and separating the coefficient of i and j we obtain respectively

$$\frac{1}{\rho}\frac{dp}{ds}+Q\left(1+\frac{dQ}{ds}\right)=0 \tag{59}$$

$$\frac{1}{\rho}\frac{dp}{dn}+Q^2\frac{d\omega}{ds}=0 \tag{60}$$

These are the equations of momentum in the present case.

From (58), (59) and (60) we obtain

$$\frac{dp}{ds}=-\rho Q\left(1+\frac{dQ}{ds}\right)=\left(M^2-2-Q\frac{d\omega}{dn}\right)/(1-M^2) \tag{61}$$

Operating $\overset{/}{\nabla}$ on (4) and using (55) and (60) we obtain

$$\frac{d\rho}{ds}=\frac{1}{c^2}\frac{dp}{ds} \tag{62}$$

$$c^2\frac{d\rho}{dn}=-\rho Q^2\frac{d\omega}{ds}-p/Jcv\frac{dS}{dn} \tag{63}$$

Since is the inclination of streaklines, the curvatures of the streakline and its orthogonal trajectory K and K' are respectively given by

$$K = \frac{d\omega}{ds} \tag{64}$$

$$K' = -\frac{d}{dn}(\pi/2 + \omega) \tag{65}$$

Using (65) and (64) in (61) and (63) respectively we have

$$Q\frac{d\omega}{dn} + 1 = (M^2 - 1)\left(1 + \frac{dQ}{ds}\right) = 1 - QK' = -\left(\frac{M^2 - 1}{\rho Q}\right)\frac{dp}{ds} \tag{66}$$

$$c^2 \frac{d\rho}{dn} = -\rho Q K^2 - p/Jcv\,\frac{dS}{dn} \tag{67}$$

Equations (60), (61) (63) and (66) express the relations between the curvatures of the streaklines and their orthogonal trajectories, inclination of streaklines and the flow quantities p, ρ, M and Q.

§ 5. COMPATIBILITY EQUATIONS

In this section we shall obtain the compatibility conditions in α_1, ρ_1, γ_1 system, satisfied by pseduo-stationary gas flows analogous to Berker.[1]

Putting $\vec{j} = (\vec{Q}.\nabla')\vec{Q} + \vec{Q}$, in (10) and operating curl on the resultant equation we get

$$\rho\,\nabla' \wedge \vec{j} + \nabla'\rho \wedge \vec{j} = 0 \tag{68}$$

Now taking the scalar product of (68) by $\vec{j}$ we have

$$\vec{j}.\,\mathrm{curl}\,\vec{j} = 0 \tag{69}$$

Expressing the equation (68) as

$$\mathrm{curl}\,\vec{j} + \nabla' \log\rho \wedge \vec{j} = 0 \tag{70}$$

and taking the cross product of this expression by $\vec{Q}$ we have

$$\mathrm{curl}\,\vec{j} \wedge \vec{Q} = (\vec{Q}.\vec{j})\,\nabla' \log\rho - (\vec{Q}.\nabla' \log\rho)\,\vec{j} \tag{71}$$

Using (9) in (71) we get

$$\frac{\mathrm{curl}\,\vec{j} \wedge \vec{Q} - (3 + \theta)\,\vec{j}}{\mu} = \nabla \log\rho \tag{72}$$

where $\theta = \nabla'.\vec{Q}$ and $\mu = \vec{Q}.\vec{j}$

Operating curl on (72) we obain

(73) $$\text{curl}\left\{\frac{\text{curl}\,\vec{j}\wedge\vec{Q}-(3+\theta)\,\vec{j}}{\mu}\right\}=0$$

Operating $\overset{/}{\nabla}$ on state equation and multiplying the resultant expression by $\vec{Q}$ scalarly we get

(74) $$\vec{Q}\cdot\overset{/}{\nabla}\rho=\frac{\partial\rho}{\partial p}\,\vec{Q}\cdot\overset{/}{\nabla}p+\frac{\partial\rho}{\partial S}\,\vec{Q}\cdot\overset{/}{\nabla}S$$

Using the energy equation and (10) in (74) it follows

(75) $$\left(\vec{Q}\,\overset{/}{\nabla}\log\rho\right)=-\frac{\partial\rho}{\partial p}\left(\vec{Q}\cdot\vec{j}\right)$$

Using equation (9) in this and then operating $\overset{/}{\nabla}$ on the resultant relation we have

(76) $$\overset{/}{\nabla}\left(\frac{\mu}{3+\theta}\right)=\overset{/}{\nabla}\left(\frac{\partial p}{\partial\rho}\right)=\frac{\partial}{\partial\rho}\,\overset{/}{\nabla}p$$

Now multiplying the above equation by curl $\vec{j}$ scalarly and using (66) we have

(77) $$\overset{/}{\nabla}\left(\frac{\mu}{3+\theta}\right)\cdot\text{curl}\,\vec{j}=0$$

Equations (69), (73) and (77) constitute the compatibility conditions to be satisfied by a pseudo-stationary gas flow.

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SOME RESULTS INVOLVING APPELL'S FUNCTION F_4

By

P. N. RATHIE

Department of Mathematics, M. R. Engineering College, Jaipur

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ABSTRACT

Some infinite integrals involving Appell's function F_4 have been evaluated with the help of a theorem on Meijer's Bessel transform and Parseval-Goldstein theorem for Hankel transform respectively. These results have been used to express F_4 as an infinite series of F_4. Some of the interesting particular cases have also been mentioned.

1. In a recent paper [4 pp. 113–118], I have expressed the various transforms of the Appell's function F_4 in terms of an infinite series of Meijer's G-function

The object of this paper is to evaluate a few infinite integrals involving the products of Appell's function F_4 and to sum certain infinite series of F_4.

The well-known Hankel and Meijer's Bessel transforms have been defined as follows :

(1) $$j_\nu \{f(t)\,;p\} = \int_0^\infty (pt)^{\frac{1}{2}} J_\nu (pt) f(t)\, dt,\ p > 0.$$

(2) $$k_\nu \{f(t)\,;p\} = \int_0^\infty (pt)^{\frac{1}{2}} K_\nu (pt) f(t)\, dt,\ R(p) > 0.$$

2. The first result to be proved is

(3) $$\int_0^\infty x^{2\nu-1}\, F_4\left[\alpha, \beta\,; \sigma, \nu\,; \frac{b^2}{d^2}, -\frac{x^2}{d^2}\right] F_4\left[\gamma, \delta\,; \mu, \nu\,; -\frac{a^2}{c^2}, -\frac{x^2}{c^2}\right] dx.$$

$$= \frac{\Gamma(\sigma)\,[\Gamma(\nu)]^2}{2\,\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\nu)\,\Gamma(\delta)} \sum_{r=0}^{\infty} \sum_{\alpha,\beta} \frac{b^{2r} d^{2\beta}\,\Gamma(\alpha-\beta)\,\Gamma(\gamma+\beta-\nu+2r)\,\Gamma(\delta+\beta-\nu+2r)}{\Gamma(r+1)\,\Gamma(\sigma+r)\, c^{2r+2\beta-2\nu}}$$

$$F_4\left[\gamma+\beta-\nu+2r, \delta+\beta-\nu+2r\,; \mu, \beta-\alpha+1\,; -\frac{a^2}{c^2}, \frac{d^2}{c^2}\right],$$

for $R(\alpha+\gamma-\nu) > 0$, $R(\alpha+\delta-\nu) > 0$, $R(\beta+\delta-\nu) > 0$, $R(\beta+\gamma-\nu) > 0$, $R(c) > 0$, $R(d) > 0$, $R(\nu) > 0$, $R(c) > |\,Im(a)\,| + |\,R(d)\,|$.

Proof : Applying Parseval-Goldstein theorem for Hankel transform in [6, p. 38].

$$j_\nu \{x^{\lambda-3/2} J_\mu(ax)\, K_\rho(cx)\,; p\}$$

$$= \frac{2^{\lambda-2} a^\mu\, \Gamma\{\frac{1}{2}(\lambda+\mu+\nu\pm\rho)\}\, p^{\nu+\frac{1}{2}}}{c^{\lambda+\mu+\nu}\,\Gamma(\mu+1)\,\Gamma(\nu+1)}$$

$$F_4\left[\tfrac{1}{2}(\lambda+\mu+\nu-\rho),\ \tfrac{1}{2}(\lambda+\mu+\nu+\rho),\ \mu+1,\ \nu+1\ ;\ -\frac{a^2}{c^2},\ -\frac{p^2}{c^2}\right],$$

for $R(c) > |\,Im(a)\,| + |\,Im(p)\,|$, $R(\lambda+\mu\pm\rho+\nu) > 0$.

and [2, p. 110]

$$j\nu\left[x^{\nu+1/2}\ F_4\left\{\tfrac{1}{2}(k+\sigma-\delta+\nu),\ \tfrac{1}{2}(k+\sigma+\delta+\nu)\ ;\ 1+\sigma,\ 1+\nu\ ;\ \frac{b^2}{d^2},\ \frac{x^2}{d^2}\right\};p\right]$$

$$=\frac{d^{k+\nu+\sigma}\ \Gamma(1+\nu)\ \Gamma(1+\sigma)\ p^{k-\frac{3}{2}}}{2^{k-2}\ b^{\sigma}\ \Gamma\{\tfrac{1}{2}(k+\sigma+\nu\pm\delta)\}}\ K_{\delta}\,(pd)\ I_{\sigma}\,(bp),$$

for $R(\nu+1) > 0$, $R(k+\sigma\pm\delta-\tfrac{1}{2}) > 0$,

writing $\displaystyle\sum_{r=0}^{\infty}\frac{(bt/2)^{\sigma+2r}}{\Gamma(r+1)\ \Gamma(\sigma+1+r)}$ for $I_{\sigma}\,(bt)$;

interchanging the order of integration and summation, we have

$$\int_0^{\infty} x^{2\nu+1}\ F_4\left[\tfrac{1}{2}(k+\sigma-\delta+\nu),\ \tfrac{1}{2}(k+\sigma+\delta+\nu)\ ;\ 1+\sigma,\ 1+\nu\ ;\frac{b^2}{d^2},-\frac{x^2}{d^2}\right]$$

$$F_4\left[\tfrac{1}{2}(\lambda+\mu+\nu-\rho),\ \tfrac{1}{2}(\lambda+\mu+\nu+\rho)\ ;\ \mu+1,\ \nu+1\ ;-\frac{a^2}{c^2},-\frac{x^2}{c^2}\right]dx$$

$$=\frac{c^{\lambda+\mu+\nu}\ d^{k+\nu+\sigma}\ [\Gamma(1+\nu)]^2\ \Gamma(\mu+1)\ \Gamma(\sigma+1)}{2^{k+\lambda-4}\ b^{\sigma}\ a^{\mu}\ \Gamma\{\tfrac{1}{2}(k+\sigma+\nu\pm\delta)\}\ \Gamma\{\tfrac{1}{2}(\lambda+\mu+\nu\pm\rho)\}}$$

$$\sum_{r=0}^{\infty}\frac{(b/2)^{\sigma+2r}}{\Gamma(r+1)\ \Gamma(\sigma+1+r)}\int_0^{\infty} t^{k+\lambda+\sigma+2r-3}\ J_{\mu}\,(at)\ K_{\rho}(ct)\ K_{\delta}\,(dt)\ dt.$$

Now evaluating the t–integral by Bailey's result [2, p. 110] we get (3) on relacing $\tfrac{1}{2}(k+\sigma+\nu-\delta)$ by α, $\tfrac{1}{2}(k+\sigma+\nu+\delta)$ by β, $\tfrac{1}{2}(\lambda+\mu+\nu-\rho)$ by γ, $\tfrac{1}{2}(\lambda+\mu+\nu+\rho)$ by δ, $\sigma+1$ by σ, $\mu+1$ by μ, and $\nu+1$ by ν.

When $b\to 0$, (3) yield the result (2, p. 111) recently given by Sharma.

3. The second result to be prove is

$$(4)\qquad \int_0^{\infty} x^{2\nu-1}\ F_4\left[\alpha,\ \beta\ ;\ \sigma,\ \nu\ ;-\frac{b^2}{d^2},-\frac{x^2}{d^2}\right]$$

$$F_4\left[\nu+\sigma-\beta,\ \nu+\sigma-\alpha\ ;\ \sigma,\ \ \nu\ ;-\frac{a^2}{c^2},-\frac{x^2}{c^2}\right]dx$$

$$=\frac{2^{2\nu+2\sigma-\alpha-\beta}\ d^{\alpha+\beta}\ \Gamma(\sigma)\ [\Gamma(\nu)]^2}{2\,\Gamma(\alpha)\ \Gamma(\beta)\ \Gamma(\sigma+\nu-\alpha)\ \Gamma(\sigma+\nu-\beta)}$$

$$\sum_{\alpha,\beta} \frac{(cd)^{\beta-\alpha}\,\Gamma(\alpha-\beta)\,\Gamma(\beta-\alpha+\sigma)}{(a^2+b^2+c^2+d^2)^{\beta-\nu+\sigma}}$$

$$F_4\left[\frac{\beta-\alpha+\sigma}{2}, \frac{\beta-\alpha+\sigma+1}{2}, \beta-\alpha+1, \sigma; \frac{4c^2\,d^2}{(a^2+b^2+c^2+d^2)^2}, \frac{4a^2\,b^2}{(a^2+b^2+c^2+d^2)^2}\right],$$

for $R(\alpha+\sigma-\beta)>0, R(\beta+\sigma-\alpha)>0, R(\sigma)>0, R(\nu)>0, R(c)>0, R(d)>0.$

Proof. : Taking

$$f(x)=x^{-\lambda+\frac{5}{2}}K_\mu\,(cx)\,J_\rho\,(bx),$$

we have [6, p. 38]

$$j_\nu\left[\,x^{-\lambda+\frac{5}{2}}\,K_\mu(cx)\,J_\rho(bx)\,;y\,\right]$$

$$=\frac{2^{-\lambda+2}\,y^{\nu+\frac{1}{2}}\,b^\rho\,\Gamma\{\frac{1}{2}(-\lambda+4+\nu+\rho\pm\mu)\}}{c^{-\lambda+4+\nu+\rho}\,\Gamma(\nu+1)\,\Gamma(\rho+1)}$$

$$F_4\left[\tfrac{1}{2}(-\lambda+4+\nu+\rho-\mu), \tfrac{1}{2}(-\lambda+4+\nu+\rho+\mu)\,;\nu+1, \rho+1\,;-\frac{y^2}{c^2}, -\frac{b^2}{c^2}\right],$$

for $R(-\lambda+4+\nu+\rho\pm\mu)>0, R(c)>|\,I_m(a)\,|+|\,I_m(b)\,|$.

Hence using the following theorem due to Sharma [2, p. 109], if $f(x)$ and j_ν $\{f(x)\,;y\}$ both belong to $L(0,\infty)$, and if $R(\nu+\frac{1}{2})>0$, $R(\lambda\pm\mu+\nu+\rho)>0$, $R(p)>0$, and a is real, then

(5) $$k_\mu\,\{x^{\lambda-2}\,J_\rho(ax)\,f(x)\,;p\}=\frac{\Gamma\,\{\frac{1}{2}(\lambda\pm\mu+\nu+\rho)\}}{\Gamma(1+\nu)\,\Gamma\,(1+\rho)}$$

$$2^{\lambda-2}\,a^\rho\,p^{\frac{1}{2}-\lambda-\nu-\rho}\int_0^\infty y^{\nu+\frac{1}{2}}j_\nu\,\{f(x)\,;y\}$$

$$F_4\left[\,\tfrac{1}{2}(\lambda-\mu+\nu+\rho), \tfrac{1}{2}\,(\lambda+\mu+\nu+\rho)\,;1+\rho, 1+\nu\,;-\frac{a^2}{p^2}, -\frac{y^2}{p^2}\right]dy$$

we find that

$$\int_0^\infty y^{2\nu+1}F_4\left[\tfrac{1}{2}(-\lambda+4+\nu+\rho-\mu), \tfrac{1}{2}(-\lambda+4+\nu+\rho+\mu)\,;\nu+1, \rho+1\,;-\frac{y^2}{c^2}, -\frac{b^2}{c^2}\right]$$

$$F_4\left[\tfrac{1}{2}(\lambda+\nu+\rho-\mu), \tfrac{1}{2}(\lambda+\nu+\rho+\mu)\,;\rho+1, \nu+1\,;-\frac{a^2}{p^2}, -\frac{y^2}{p^2}\right]dy$$

$$=\frac{c^{-\lambda+4+\nu+\rho}\,p^{\lambda+\rho+\nu}\,[\Gamma(\nu+1)\,\Gamma(\rho+1]^2}{(ab)^\rho\,\Gamma\{\frac{1}{2}(-\lambda+4+\nu+\rho\pm\mu)\}\,\Gamma\{\frac{1}{2}(\lambda+\nu+\rho\pm\mu)\}}$$

$$\int_0^\infty x\,K_\mu(px)\,K_\mu(cx)\,J_\rho\,(ax)\,J_\rho\,(bx)\,dx.$$

Now evaluating the integral on the right by [5, p. 131]

$$(6) \qquad \int_0^\infty t\, K_\nu(\alpha t)\, K_\nu(\beta t)\, J_\mu(\gamma t)\, J_\mu(\delta t)\, dt.$$

$$= \frac{(\gamma\delta)^\mu}{2\Gamma(\mu+1)} \sum_{\nu, -\nu} \frac{(\alpha\beta)^\nu\, \Gamma(-\nu)\, \Gamma(\mu+\nu+1)}{(\alpha^2+\beta^2+\gamma^2+\delta^2)^{\mu+\nu+1}}$$

$$F_4\left[\frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2}; \nu+1, \mu+1; \frac{4\alpha^2\beta^2}{(\alpha^2+\beta^2+\gamma^2+\delta^2)^2}, \frac{4\gamma^2\delta^2}{(\alpha^2+\beta^2+\gamma^2+\delta^2)^2}\right];$$

for $R(1+\mu\pm\nu) > 0$, $R(\alpha+\beta) > |Im(\gamma)| + |Im(\delta)|$,
replacing $\frac{1}{2}(-\lambda+4+\nu+\rho-\mu)$ by α, $\frac{1}{2}(-\lambda+4+\nu+\rho+\mu)$ by β, $\nu+1$ by γ, $\rho+1$ by σ, c by d and p by c, we arrive at (4).

4. Taking $b=ib$, $\gamma=\nu+\sigma-\beta$, $\delta=\nu+\sigma-\alpha$ and $\mu=\sigma$ in (3) and comparing with (4), we obtain

$$(7) \qquad \sum_{r=0}^{\infty} \frac{(-1)^r b^{2r}\, \Gamma(\sigma+\beta-\alpha+2r)\, \Gamma(\sigma+2r)}{\Gamma(r+1)\, \Gamma(\sigma+r)\, c^{2r+2\beta-2\alpha+2\sigma}}$$

$$F_4\left[\sigma+2r, \sigma+\beta-\alpha+2r; \sigma, \beta-\alpha+1; -\frac{a^2}{c^2}, \frac{d^2}{c^2}\right]$$

$$= \frac{\Gamma(\beta-\alpha+\sigma)}{(a^2+b^2+c^2+d^2)^{\beta-\alpha+\sigma}} F_4\left[\frac{\beta-\alpha+\sigma}{2}, \frac{\beta-\alpha+\sigma+1}{2}; \beta-\alpha+1, \sigma; \frac{4c^2d^2}{(a^2+b^2+c^2+d^2)^2}, \frac{4a^2b^2}{(a^2+b^2+c^2+d^2)^2}\right]$$

for $R(c) > |R(d)| + |Im(a)|$, $R(\beta-\alpha+\sigma) > 0$, $R(\sigma) > 0$, $R[(a\pm b)^2] + R[(c\pm d)^2] > 0$.

As $b \to 0$, (7) reduces to

$$(8) \qquad F_4\left[\sigma, \sigma+\beta-\alpha; \sigma, \beta-\alpha+1; -\frac{a^2}{c^2}, \frac{d^2}{c^2}\right]$$

$$= \frac{c^{2\beta-2\alpha+2\sigma}}{(a^2+c^2+d^2)^{\beta-\alpha+\sigma}}\, {}_2F_1\left[\frac{\beta-\alpha+\sigma}{2}, \frac{\beta-\alpha+\sigma+1}{2}; \beta-\alpha+1\; \frac{4c^2d^2}{(a^2+c^2+d^2)^2}\right]$$

Which on applying [1, p. 113 (34))] gives a known result [1, p. 238 (8)].

On using [7, p. 102]

$$\sum_{r=0}^{\infty} \frac{\lfloor n\, (\alpha+\beta+1)_n\, t^n}{(\alpha+1)_n\, (\beta+1)_n} P_n^{(\alpha,\beta)}(\cos 2\phi)\, P_n^{(\alpha,\beta)}(\cos 2\psi)$$

$$= (1+t)^{-\alpha-\beta-1} F_4\left[\tfrac{1}{2}(\alpha+\beta+1), \tfrac{1}{2}(\alpha+\beta+2); \alpha+1, \beta+1; \frac{a^2}{k^2}, \frac{b^2}{k^2}\right],$$

Where $a = \sin\phi \sin\psi$, $b = \cos\phi \cos\psi$, $k = \frac{1}{2}(t^{-\frac{1}{2}} + t^{\frac{1}{2}})$ and

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n \, \Gamma(n+\alpha+1)}{\lfloor n \; \Gamma(\alpha+1)} \; {}_2F_1\left(-n, n+\alpha+\beta+1 \, ; \alpha+1 \, ; \frac{1-x}{2}\right);$$

(7) becomes

$$(9) \qquad (2t^{1/2})^{\beta-\alpha+\sigma} \sum_{r=0}^{\infty} \frac{\lfloor n \; \Gamma(\beta-\alpha+\sigma+n) t^n}{(\beta-\alpha+1)_n (\sigma)_n} P_n^{(\beta-\alpha,\, \sigma-1)}(\cos 2\phi) \, P_n^{(\beta-\alpha,\, \sigma-1)}(\cos 2\psi)$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \, b^{2r} \, \Gamma(\sigma+\beta-\alpha+2r) \, \Gamma(\sigma+2r)}{\Gamma(r+1) \, \Gamma(\sigma+r) \, c^{2r+2\beta-2\alpha+2\sigma}}$$

$$F_4\left[\sigma+2r, \sigma+\beta-\alpha+2r \, ; \sigma, \beta-\alpha+1 \, ; -\frac{a^2}{c^2}, \frac{d^2}{c^2}\right],$$

where $2cd = \sin\phi \sin\psi$, $2ab = \cos\phi \cos\psi$, $a^2+b^2+c^2+d^2 = \frac{1}{2}(t^{-\frac{1}{2}} + t^{\frac{1}{2}})$, $R(c) > |I_m(a)| + |R(d)|$, $R(\beta-\alpha+\sigma) > 0$, $R(\sigma) > 0$.

Taking $d = ia$ in (7) and using [3, p. 57], we get

$$(10) \qquad F_4\left[\frac{\beta-\alpha+\sigma}{2}, \frac{\beta-\alpha+\sigma+1}{2} \, ; \beta-\alpha+1, \sigma \, ; -\frac{4a^2c^2}{(b^2+c^2)^2}, \frac{4a^2b^2}{(b^2+c^2)^2}\right]$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \, b^{2r} \, (b^2+c^2)^{\beta-\alpha+\sigma} (\beta-\alpha+\sigma)_{2r} \, (\sigma+r)_r}{\lfloor r \; c^{2r+2\beta-2\alpha+2\sigma}}$$

$${}_4F_3\left[\sigma+2r, \sigma+\beta-\alpha+2r, \frac{\sigma+\beta-\alpha}{2}, \frac{\sigma+\beta-\alpha+1}{2} \, ; \sigma, \beta-\alpha+1, \sigma+\beta-\alpha \, ; -\frac{4a^2}{c^2}\right],$$

for $R(c) > 2\,|I_m(a)|$, $R(\sigma) > 0$, $R(\sigma+\beta-\alpha) > 0$.

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POLARIMETRIC ESTIMATION OF TUNGSTEN

By

O. N. PERTI

Chemical Laboratories, M. N. R. Engg. College, Allahabad

and

I. PRAKASH

Central Institute of Education, Delhi

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ABSTRACT

Tungsten in the form of sodium tungstate can be estimated polarimetrically by precipitating the salt from its aqueous solution in the form of optically active compound, using quinine sulphate solution as precipitant. The precipitate is then dissolved in a mixture of dioxan, ethanol and dilute hydrochloric acid, and polarimetric estimation is carried out by employing an empirical equation of the type

$$W = 37 \cdot 6 \ R - 0 \cdot 249 \text{ mg of W (VI)/50 ml}$$

where W and R denote weight of metallic ion in mg/50 ml, and R is angle of rotation.

In previous communications we have described polarimetric estimation of cadmium[1], bismuth[2], vanadium[3]. copper[4], silver[5], cobalt[6] and palladium[7]. In the present paper estimation of tungsten has been described on similar lines.

Brucine[8] has been used for gravimetric estimation of tungsten in the form of a complex of the alkaloid and tungstic acid, and igniting the latter to WO_3. Cinchonine[9] and quinine[10] have also been used for gravimetric estimation of tungsten.

EXPERIMENTAL PROCEDURE

Following solutions were prepared :

(*i*) 0·8968g of sodium tungstate ($Na_2WO_4.2H_2O$ – B.D.H sample) was dissolved in 50 ml of distilled water.

(*ii*) 12g of quinine sulphate was dissolved in 500 ml of distilled water having a few drops of sulphuric acid.

(*iii*) 4N hydrochloric acid.

To an aliquot of sodium tungstate solution, aqueous solution of quinine sulphate was added in excess. For quantitative precipitation of W (VI), quinine sulphate solution equal to one half of volume of salt solution was used. White precipitate obtained was slightly warmed and left in the beaker for a couple of minutes at room temperature. It was then cooled and filtered. It was washed with a few ml of cold water and was then dissolved in dioxen, 4N HCl and alcohol (8 : 1 : 1) mixture. Volume of solution was made to 50 ml. The solution thus obtained was used for polarimetric measurement at room tempreture (14°C)

In Table 1 volume of solution taken corresponding to different amounts of W(VI) ions present are given, along with their optical rotatory power, as gravimetrically estimated using 8–hydroxy quinoline[11]. It is this estimated amount which is given in the table.

TABLE 1

Optical rotation of solutions containing different amounts of W (VI) *in the form of optically active complex*

Serial No.	W (VI/50 ml (mg)	Optical rotation in degrees for Hg 5461
1.	2·63	0·07
2.	5·25	0·15
3.	7·88	0·22
4.	10·50	0·28
5.	13·13	0·36
6.	15·75	0·43
7.	21·00	0·56

When a graph was plotted between concentrations of W(VI) mg/50 ml and optical rotation of resulting complex, a straight line was obtained. The equation of this straight line was calculated by the method of least squares. It is given below

$$W = 37.6R - 0{\cdot}249 \text{ mg. of } W(VI)/50 \text{ ml.}$$

where W is the weight of W(VI) ions per 50 ml in mg and R is the angle of rotation in degrees.

In Table 2 a comparison is made between gravimetrically estimated amounts of W(VI) ions and values calculated on the basis of the empirical equation given above.

TABLE 2

A comparison between gravimetrically estimated and polarimetrically determined values of W(VI) *in different volumes of salt solution*

Amount of W(VI) in mg

Serial No.	Gravimetrically estimated (*a*)	Polarimetrically estimated (*b*)	Difference (*a*) – (*b*) mg.
1.	2·63	2·38	+ 0·25
2.	2·25	5·39	– 0·14
3.	7·88	8·02	– 0·14
4.	10·50	10·28	+ 0·22
5.	13·13	13·29	– 0·16
6.	15·75	15·92	– 0·17
7.	21·00	20·81	+ 0·19

DISCUSSION

It is clear from Table 2 that polarimetric estimation of W(VI) gives fairly accurate results in concentrations ranging from 5·25 mg. to 21·00 mg. of W(VI) per 50 ml. This method can be applied in routine estimations where great accuracy is not desired. Polarimetric estimation compares favourably with colorimetric determinations. The accuracy of the polarimetric method can be further increased by employing a larger polarimeter tube wherever possible and using photoelectric methods of measurement of optical rotation in the ultraviolet where the rotation values of the complex is usually high.

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ON SOME INTEGRALS INVOLVING JACOBI POLYNOMIALS

By

R. C. VARMA

Department of Mathematics, Government Engineering College, Jabalpur

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ABSTRACT

The expansion of x^n has been explicitly found out in terms of Jacobi Polynomials. The integral

$$\int_{-1}^{1} x^s(1-x)^\rho (1+x)^\sigma P_n^{(\alpha,\beta)}(x)\, dx$$

has also been evaluated for $s > n$. Three special cases of this also occur with $\rho = \alpha$; $\sigma = \beta$ and $\rho = \alpha, \sigma = \beta$.

1. INTRODUCTORY

The expansion of x^n is given in terms of Legendre and Gegenbauer Polynomials [1, p. 181 ; p. 283], but it appears that it has not been explicitly given in terms of Jacobi Polynomials. The object of this paper is to obtain an expansion of x^n in terms of Jacobi Polynomials and to utilise the result to evaluate certain integrals involving Jacobi Polynomials which are believed to be new.

2. FORMULAE REQUIRED IN THE PROOF

We have the following results [1, p. 254, p. 257, p. 260, p. 262].

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\ {}_2F_1\left[\begin{matrix}-n,\ 1+\alpha+\beta+n\ ;\\ 1+\alpha\ ;\end{matrix}\ \frac{1-x}{2}\right]. \tag{2·1}$$

$$(1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n\, n!} D^n \left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right]. \tag{2·2}$$

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta \left[P_n^{(\alpha,\beta)}(x)\right]^2 dx = \frac{2^{1+\alpha+\beta}\,\Gamma(1+\alpha+n)\,\Gamma(1+\beta+n)}{n!\,(1+\alpha+\beta+2n)\,\Gamma(1+\alpha+\beta+n)}. \tag{2·3}$$

and

$$(1-x)^n = 2^n(1+\alpha)_n \sum_{k=0}^{n} \frac{(-n)_k\,(1+\alpha+\beta+2k)\,(1+\alpha+\beta)_k}{(1+\alpha+\beta+k)_{n+k+1}\,(1+\alpha)_k}\, P_k^{(\alpha,\beta)}(x). \tag{2·4}$$

3. Bhonsle[3] has obtained the following integral for $s = n$, where s is a non-negative integer.

$$\int_{-1}^{1} x^s(1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x)\, dx = \frac{2^{n+\alpha+\beta+1}\,\Gamma(1+\alpha+n)\,\Gamma(1+\beta+n)}{\Gamma(2+\alpha+\beta+2n)} \tag{3·1}$$

and for $s < n$, the integral is zero, [1, p. 261].

For values of $s > n$, by expressing

$$x^{s-n} = \left\{\frac{1+x}{2} - \frac{1-x}{2}\right\}^{s-n}$$

We evaluate the following integral,

$$\int_{-1}^{1} x^s(1-x)^\alpha \ (1+x)^\beta \ P_n^{(\alpha,\beta)}(x) \ dx$$

$$= \frac{2^{n+\alpha+\beta+1} \ s \,! \ \Gamma(1+\alpha+n) \ \Gamma(1+\beta+s)}{n\,! \ (s-n)\,! \ \Gamma(n+s+\alpha+\beta+2)} \ {}_2F_1 \left[\begin{matrix} -s+n, \ 1+\alpha+n \, ; \\ -\beta-s \, ; \end{matrix} -1\right]. \quad (3{\cdot}2)$$

Or if we express, $x^{s-n} = \{1-(1-x)\}^{s-n}$,

$$\int_{-1}^{1} x^s(1-x)^\alpha \ (1+x)^\beta \ P_n^{(\alpha,\beta)}(x) \ dx$$

$$= \frac{2^{n+\alpha+\beta+1} \ s\,! \ \Gamma(1+\alpha+n) \ \Gamma(1+\beta+n)}{n\,! \ (s-n)\,! \ \Gamma(2+\alpha+\beta+2n)} \ {}_2F_1 \left[\begin{matrix} -s+n, \ 1+a+n \, ; \\ 2+\alpha+\beta+2n \, ; \end{matrix} \ 2\right]. \quad (3{\cdot}3)$$

The result [3·3) can also be obtained from (3·2) by hypergeometric transformation. In (3·2) and (3·3) if we put $s = n$, we get (3·1).

4. EXPANSION OF x^n IN JACOBI POLYNOMIALS

If n be a non-negative integer, we can have the expansion [3, p. 147],

$$x^n = a_0 \ P_n^{(\alpha,\beta)}(x) + a_1 \ P_{n-1}^{(\alpha,\beta)}(x) + \ldots + a_r \ P_{n-r}^{(\alpha,\beta)}(x) + \ldots + a_n \quad (4{\cdot}1)$$

where,

$$a_r = \frac{\int_{-1}^{1} x^n(1-x)^\alpha \ (1+x)^\beta \ P_{n-r}^{(\alpha,\beta)}(x) \ dx}{\int_{-1}^{1} (1-x)^\alpha \ (1+x)^\beta \left[P_{n-r}^{(\alpha,\beta)}(x)\right]^2 dx}. \quad (4{\cdot}2)$$

From (3·3) and (2·3), we have

$$a_r = \frac{n\,! \ 2^{n-r} \ \Gamma(1+\alpha+\beta+n-r)}{r\,! \ \Gamma(1+\alpha+\beta+2n-2r)} \ {}_2F_1 \left[\begin{matrix} -r, \ 1+\alpha+n-r \, ; \\ 2+\alpha+\beta+2n-2r \, ; \end{matrix} \ 2\right] \quad (4{\cdot}3)$$

Therefore, (4·1) can be expressed as

$$x^n = \sum_{r=0}^{n} a_r \ P_{n-r}^{(\alpha,\beta)}(x). \quad (4{\cdot}4)$$

$$= \sum_{r=0}^{n} \frac{n\,! \ 2^{n-r} \ \Gamma(1+\alpha+\beta+n-r)}{r\,! \ \Gamma(1+\alpha+\beta+2n-2r)} \ {}_2F_1 \left[\begin{matrix} -r, \ 1+\alpha+n-r \, ; \\ 2+a+\beta+2n-2r \, ; \end{matrix} \ 2\right] P_{n-r}^{(\alpha,\beta)}(x). \quad (4{\cdot}5)$$

Again expressing, $x^n = \{1-(1-x)\}^n$
and using (2·4) we obtain

$$x^n = \sum_{k=0}^{n} \frac{n!\,2^k\ \Gamma(1+\alpha+\beta+k)}{(n-k)!\ \Gamma(1+\alpha+\beta+2k)}\ {}_2F_1\left[\begin{matrix}-n+k,\ 1+\alpha+k\ ;\\ 2+\alpha+\beta+2k\ ;\end{matrix}\ 2\right] P_k^{(\alpha,\beta)}(x) \quad (4\cdot6)$$

We can also get (4·6) from (4·5) by writing the series (4·5) in reverse order.

5. INTEGRALS

We expand $P_n^{(\alpha,\beta)}(x)$ with the help of (2·1) and integrate term by term so as to obtain

$$\int_{-1}^{1} x^s(1-x)^\rho\,(1+x)^\sigma\ P_n^{(\alpha,\beta)}(x)\,dx$$

$$= \frac{2^{\rho+\sigma+1}(1+\alpha)_n\ \Gamma(\sigma+1)\ \Gamma(\rho+1)}{n!\ \Gamma(\rho+\sigma+2)} \sum_{r=0}^{n} \frac{(-n)_r\,(1+\alpha+\beta+n)_r\,(1+\rho)_r}{r!\,(1+\alpha)_r\,(2+\rho+\sigma)_r} \times$$

$$\times\ {}_2F_1\left[\begin{matrix}-s,\ r+\rho+1\ ;\\ r+\rho+\sigma+2\ ;\end{matrix}\ 2\right]. \quad (5\cdot1)$$

where $s > n$.

As the special cases of the above, the following integrals,

$$\int_{-1}^{1} x^s\,(1-x)^\alpha\,(1+x)^\sigma\ P_n^{(\alpha,\beta)}(x)\,dx, \quad (5\cdot2)$$

and

$$\int_{-1}^{1} x^s(1-x)^\rho\,(1+x)^\beta\ P_n^{(\alpha,\beta)}(x)\,dx, \quad (5\cdot3)$$

can also be obtained by substituting in (5·1) $\rho = \alpha$ and $\sigma = \beta$ respectively.

By expressing $x^s = \{(1+x)-1\}^s = (-1)^s\{1-(1+x)\}^s$ and expanding it by means of binomial theorem, and making use of (2·2) and integrating by parts, the integral (5·2) can be obtained in a little more compact form as follows:

$$\int_{-1}^{1} x^s(1-x)^\alpha\,(1+x)^\sigma\ P_n^{(\alpha,\beta)}(x)\,dx$$

$$= \frac{(-1)^n}{2^n\,n!} \int_{-1}^{1} (-1)^s\,\{1-(1+x)\}^s\,(1+x)^{\sigma-\beta}\ D^n\{(1-x)^{n+\alpha}\,(1+x)^{n+\beta}\}\,dx$$

$$= \frac{(-1)^{n+s}}{2^n\,n!} \sum_{r=0}^{s} \frac{(-s)_r}{r!} \int_{-1}^{1} (1+x)^{r+\sigma-\beta}\ D^n\ (1-x)^{n+\alpha}\,(1+x)^{n+\beta}\}\,dx$$

$$= \frac{(-1)^s}{2^n\ n!} \sum_{r=0}^{s} \frac{(-s)_r\ \Gamma(r+\sigma-\beta+1)\ 2^{n+\alpha+\sigma+r+1}\ \Gamma(n+\alpha+1)\ \Gamma(r+\sigma+1)}{r!\ \Gamma(r+\sigma-\beta-n+1)\ \Gamma(n+\alpha+r+\sigma+2)}$$

$$= \frac{(-1)^s\,2^{1+\alpha+\sigma}\ \Gamma(1+\alpha+n)\ \Gamma(\sigma-\beta+1)\ \Gamma(\sigma+1)}{n!\ \Gamma(\sigma-\beta-n+1)\ \Gamma(n+\alpha+\sigma+2)}\ {}_3F_2\left[\begin{matrix}-s,\ \sigma-\beta+1,\ \sigma+1\ ;\\ \sigma-\beta-n+1,\ n+\alpha+\sigma+2\ ;\end{matrix}\ 2\right] \quad (5\cdot4)$$

Similarly, by expressing, $x^s = \{1-(1-x)\}^s$ and making use of (2·2) and integrating by parts, we obtain,

$$\int_{-1}^{1} x^s(1-x)^\rho \ (1+x)^\beta \ P_n^{(\alpha,\beta)}(x)\ dx$$

$$= \frac{(-1)^n\ 2^{\rho+\beta+1}\ \Gamma(1+\beta+n)\ \Gamma(1+\rho-\alpha)\ \Gamma(1+\rho)}{n!\ \Gamma(1+\rho-\alpha-n)\ \Gamma(2+\rho+\beta+n)}\ {}_3F_2\left[\begin{matrix}-s,\ 1+\rho-\alpha,\ 1+\rho\ ; \\ \rho-\alpha-n+1,\ 2+n+\rho+\beta\ ;\end{matrix}\ 2\right] \quad (5·5)$$

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TRANSFORMATION OF A CERTAIN SERIES INVOLVING THE SOLUTIONS OF F-EQUATIONS AND CONFLUENT HYPERGEOMETRIC FUNCTIONS

By

B. M. AGRAWAL

Government Science College, Gwalior, M. P.

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ABSTRACT

In this note some transformations of a certain series involving the solutions of F-equations

$$\partial F(z,\alpha)/\partial z = F(z, \alpha+1)$$

and

$$\partial \Upsilon(z,\alpha)/\partial z = \Upsilon(z, \alpha-1)$$

and the confluent Hypergeometric Functions have been deduced. These transformations in certain cases turns out to convert an infinite series of product of two functions into finite one, thus expressing the sum of the infinite series in finite terms.

INTRODUCTION

In this note some transformations of a certain series involving the solutions of Truesdell F-equations :

$$\partial F(z, \alpha)/\partial z = F(z, \alpha+1) \tag{1}$$

$$\partial \Upsilon(z, \beta)/\partial z = \Upsilon(z, \beta+1) \tag{2}$$

and the Confluent Hypergeometric Functions have been deduced. These transformations in certain cases turns out to convert an infinite series of product of two functions into a finite one, thus expressing the sum of the infinite series in finite terms.

2. First we shall prove the following identities :

$$\sum_{r=0}^{\infty} D_x^r \left[e^{px} F_1(x)\right] D_y^r \left[e^{qy} F_2(y)\right] z^r/r\,! \tag{3}$$

$$= \exp\,[px + qy + pqz] \sum_{r=0}^{\infty} D_x^r\; F_1(x + zq)\, D_y^r\; F_2(y + zp) \cdot \frac{z^r}{r!}$$

$$\sum_{r=0}^{\infty} F(x, \alpha + r)\, D_y^r \left(e^{qy} F_2(y)\right) z^r/r\,! \tag{4}$$

$$= \exp\,(qy) \sum_{r=0}^{\infty} F(x + zq, \alpha + r)\, D_y^r\, F_2\,(y)\, z^r/r\,!$$

$$\sum_{r=0}^{\infty} \Upsilon\,(x, \beta - r)\, D_y^r \left(e^{qy} F_2(y)\right) z^r/r\,! \tag{5}$$

$$= \exp\,(qy) \sum_{r=0}^{\infty} \Upsilon(x + zq, \beta - r)\, D_y^r \left(e^{qy} F_2(y)\right) z^r/r\,!$$

where $F(x, \alpha)$ and $Y(x,\beta)$ satisfy the equations (1) and (2), and D_x denotes the differentiation with respect to x alone. These results are formal and subject to the convergence of the both sides.

Proof. Consider,

$$\sum_{r=0}^{\infty} D_x^r \left[e^{px} F_1(x) \right] D_y^r \left[e^{py} F_2(y) \right] z^r/r!$$

$$= \sum_{r=0}^{\infty} \sum_{n=0}^{r} D_x^r \left[e^{px} F_1(x) \right] D_y^{r-n} F_2(y).\ q^n e^{qy} \frac{z^r}{r!(r-n)!}$$

$$= e^{qy} \sum_{r=0}^{\infty} D_x^r \left[e^{px+pqz} F_1(x+zq) \right] D_y^r F_2(y) z^r/r!$$

$$= e^{qy+px+pqz} \sum_{r=0}^{\infty} \sum_{n=0}^{r} p^n D_x^{r-n} F_1(x+zq) D_y^r F_2(y) z^r/r!$$

$$= \exp(px+qy+pqz) \sum_{r=0}^{n} D_x^r F_1(x+zq) D_y^r F_2(y+zp) z^r/r!$$

hence proved.

On putting $p = 0$ and $F_1(x)$ as the solutions of equations (1) and (2), we obtain (4) and (5).

Now we shall add a few examples.

Example 1.

Let $F_1(x) = {}_1F_1[a, c; x]$, $F_2(y) = {}_1F_1[b, d; y]$ and $p = q = -1$.

Then on substitution in (3) we get [2, p. 344]

$$\sum_{r=0}^{\infty} \frac{(c-a)_r\ (d-b)_r}{(c)_r\ \ (d)_r}\ {}_1F_1[a, c+r; x]\ {}_1F_1[b, d+r; y]\, z^r/r!$$

$$= e^z \sum_{r=0}^{\infty} \frac{(a)_r\ (b)_r}{(c)_r\ (d)_r} F[a+r, c+r; x-z]\ F[b+r, d+r; y-z]\ z^r/r!$$

Similarly when $p = -1$, $q = -1$, we get [2, p. 344]

$$\sum_{r=0}^{\infty} \frac{(c-a)_r\ (b)_r}{(c)_r\ \ (d)_r}\ {}_1F_1[a, c+r; x]\ {}_1F_1[b+r, d+r, y-z]\, z^r/r!$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r\ (d-b)_r}{(c)_r\ \ (d)_r}\ {}_1F_1[a+r, d+r; x-z]\ {}_1F_1[b, d+r; y]\, z^r/r!$$

Example 2.

Let $F_1(x) = x^{\alpha} L_n^{\alpha}(x)$ and $F_2(y) = {}_1F_1[b, d; y]$

From (3), using [1, p. 190], we get,

$$\sum_{r=0}^{\infty} \frac{(1+n)_r (d-b)_r}{(d)_r} L_{n+r}^{\alpha-r}(x)\ {}_1F_1[b, d+r; y]\ (-z/x)^r/r!$$

$$= e^z (1-z/x)^\alpha \sum_{r=0}^{\infty} \frac{(1+n+\alpha-r)_r (b)_r}{(d)_r} L_n^{\alpha-r}(x-z)$$

$$ {}_1F_1[b+r, d+r; y-z]\ (z/x-z)^r/r! \tag{6}$$

and

$$\sum_{r=0}^{\infty} \frac{(1+n)_r (b)_r}{(d)_r} L_{n+r}^{\alpha-r}(x)\ {}_1F_1[b+r, d+r; y-z]\ (z/x)^r/r!$$

$$= (1-z/x)^\alpha \sum_{r=0}^{\infty} \frac{(1+n+\alpha-r)_r (d-b)_r}{(d)_r} L_n^{\alpha-r}(x-z)$$

$$ {}_1F_1[b, d+r; y]\ (-z/x-z)^r/r! \tag{7}$$

If we replace b by $-m$ and d by $1+\beta$ in (6) and (7), we obtain

$$\sum_{r=0}^{\infty} (1+n)_r L_{n+r}^{\alpha-r}(x)\ L_m^{\beta+r}(y)\ (-z/x)^r/r!$$

$$= e^{-z} (1-z/x)^\alpha \sum_{r=0}^{m} (1+n+\alpha-r)_r L_n^{\alpha-r}(x-z)$$

$$L_{m-r}^{\beta+r}(y-z)\ (-z/x-z)^r/r! \tag{8}$$

and

$$\sum_{r=0}^{\infty} (1+n+\alpha-r)_r L_n^{\alpha-r}(x-z)\ L_m^{\beta+r}(y)\ (-z/x-z)^r/r!$$

$$= \sum_{r=0}^{m} (1+n)_r L_{n+r}^{\alpha-r}(x)\ L_{m-r}^{\beta+r}(y-z)\ (-z/x)^r/r! \tag{9}$$

Putting $m = 0$, (8) and (9) reduce to

$$\sum_{r=0}^{\infty} (1+n)_r L_{n+r}^{\alpha-r}(x)\ t^r/r! = e^{-xt} (1+t)^\alpha L_n^\alpha [x(1+t)] \tag{10}$$

$$\sum_{r=0}^{\infty} (1+n+\alpha-r)_r L_n^{\alpha-r}(x)\ t^r/r! = L_n^\alpha [x(1+t)], \tag{11}$$

(10) reduces to a known result [1, p. 189], when $n = 0$.

Example 3.

Let $F(x, a) = x^{-\alpha} G_{p,q}^{m,n}\left(\beta x \middle/ \begin{matrix} a_1, \ldots, a_p \\ \alpha, b_2, \ldots, b_q \end{matrix}\right)$

$F_2(y) = {}_1F_1[a, c, y]$

Then from (4) on simplification we get,

$$\sum_{r=0}^{\infty} G_{p,q}^{m,n}\left(\beta x \Big/ \begin{matrix} a_1, \ldots a_p \\ r, b_2, \ldots b_q \end{matrix}\right) L_n^{\alpha+r}(y)\, t^r/r\,!$$

$$= \sum_{r=0}^{n} G_{p,q}^{m,n}\left(\beta x(1+t) \Big/ \begin{matrix} a_1, \ldots, a_p \\ r, b_2, \ldots, b_q \end{matrix}\right) L_{n-r}^{\alpha+r}(y)\,(t/1+t)^r \quad (12)$$

Similarly from (5) when,

$$\Upsilon(x, \beta) = x^{\beta-1}\, G_{p,q}^{m,n}\left(\frac{d}{x} \Big/ \begin{matrix} a_1, \ldots, \beta \\ b_1, \ldots b_q \end{matrix}\right)$$

we get,

$$\sum_{r=0}^{\infty} G_{p,q}^{m,n}\left(dx \Big/ \begin{matrix} a_1, \ldots, \beta - r \\ b_1 \ldots, b_q \end{matrix}\right) L_n^{\delta+r}(y)\, t^r/r\,!$$

$$= (1+t)^{\beta-1} \sum_{r=0}^{n} G\left(\frac{d}{1+t}\, x \Big/ \begin{matrix} a_1, \ldots, \beta - r \\ b_1, \ldots, b_q \end{matrix}\right) L_{n-r}^{\delta+r}(y)\,(t/1+t)^r/r\,! \quad (13)$$

Example 4.

Let [3, p. 21],

$$F(x, \alpha) = \psi_\alpha\,(b, x)$$

then from (4) we get,

$$\sum_{r=0}^{\infty} \psi_{\alpha+r}\,(b, x)\, L_n^{\delta+r}(y)\,(-z)^r/r\,! = \sum_{r=0}^{\infty} \psi_{\alpha+r}\,(b, x-z)\, L_{n-r}^{\delta+r}(y)\,(-z)^r/r\,! \quad (14)$$

Similarly when $F(x, \alpha) = e^x\, \psi(-\alpha\,;\, x)$ we get

$$\sum_{r=0}^{\infty} \psi_b\,(-\alpha - r\,;\, x)\, L_n^{\delta+r}(x)\,(-z)^r/r\,!$$

$$= \sum_{r=0}^{n} e^{-z}\psi_\mu(-\alpha - r\,;\, x - z)\, L_{n-r}^{\delta+r}(y)\,(-z)^r/r\,! \quad (15)$$

On putting $\alpha = n = 0$ and using the transformation,

$$\psi_n(x\,;\, a) = \psi(x\,;\, a)\, P_n(n\,;\, a)$$

(14) reduces to the known generating function of Poisson's frequency function.

$$\sum_{r=0}^{\infty} p_n(x\,;\, a)\frac{t^r}{r\,!} = e^{-z}\,(1+t/_a)^x$$

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ON GENERALIZED HYPERGEOMETRIC FUNCTIONS

By

B. M. AGRAWAL

Government Science College, Gwalior M. P.

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ABSTRACT

Kharadze (1935, 1960), Chatterjee (1955), and Sharma (1948) have given a generalisation of Legendre, Hermite, and Associated Legendre polynomial using the operator,

$$D_K^{(K-1)} = \frac{1}{z^{k-2}} \frac{d}{dz} \quad \text{and} \quad D_K^K = \frac{d}{dz} \frac{1}{z^{k-1}} \frac{d}{dz}$$

in the Rodrigues formula.

In the present paper we have deduced these generalised forms by defining a structure of numbers, from ordinary definitions of the polynomials, without using these different forms of this operator.

INTRODUCTION

Kharadze, [3, 4,] Chatterjee [2] and Sharma [5], generalized the ordinary Legendre, Hermite and Associated—Legendre polynomials by considering the operator D_K^K. They defined these functions in a generalized form according to this operator and deduced their properties. It is interesting to note that by defining a structure of numbers, one can obtain these generalized functions from ordinary functions without using the different forms of this operator. Similarly other functions can also be deduced in a generalized form and their properties can be obtained. This is the main object of this paper. In section 2, 3, 4, 5, and 6, some definitions are given and a few useful theorems have been proved. Finally in the rest of the paper the various functions have been deduced in a generalized form.

2. Let $K \neq 0$ be a fixed natural number. We define a structure

$$\ldots, -(mk+1), -mk, \ldots, (-1), 0, 1, K, K+1, 2k, \ldots, mk, mk+1, \ldots$$

which we call a field B_K. We form the reduce classes modulo as below :

$$0_K = \ldots, -mk, \ldots k, 0, k, \ldots, mk, \ldots$$

$$1_K = \ldots, -(mk+1), \ldots (-1), (K+1), \ldots (mk+1), \ldots$$

The following laws for calculating with these two spaces are constructed according to their remainders (residues), thus,

$$0_K + 0_K = 1_K + 1_K = 0_K, \; 0_K + 1_K = 1_K$$

$$0_K \,.\, 0_K = 0_K \,.\, 1_K = 0_K, \; 1_K \,.\, 1_K = 1_K$$

Thus multiplication and addition can be defined by the following tables :

$+$	mk	$mk+1$
rk	$(m+r)\,k$	$(m+r)\,k+1$
$rk+1$	$(m+r)\,k+1$	$(m+r+1)k$

x	mk	$mk+1$
rk	$2mrk$	$(2mr+r)k$
$rk+1$	$(2mr+m)k$	$(2mr+m+r)\ k+1$

We define also with the usual notations over the B_k field, the following :

$$(a)_b = \frac{\Gamma\ (a+b)}{\Gamma\ (a)} = a\ (a+1)\ (a+k)\ (a+k+1)\ \ldots .\ (a+b-1)$$

$$a\ ! = 1\ .\ K.\ K+1\ \ldots .\ a$$

$$\Gamma(a) = (a-1)\ \Gamma(a-1) = K^a \int_{0K}^{\infty} e^{-kt}\ t^{a-1}\ dt$$

where a, b both $\varepsilon\ B_k$. The last terms in the above expensions can be derived according to the above tables. S_K is the inverse of the operator D_k, which we shall define later on. Then.

$${}_KF_1\left[\begin{matrix} a, b \\ c \end{matrix}\ z\right] = \sum_{\substack{r=0 \\ r\varepsilon B_K}}^{\infty} \frac{(a)_r\ (b)_r}{(c)_r\ r\,!}\ z^r = 1 + \frac{a.\ b}{c.\ 1}\ z + \frac{a(a+1).\ b(b+1)}{c(c+1)\ 1.\ K}\ z^K + \ldots .$$

$$E_K\ (z) = 1 + \frac{z}{1} + \frac{z^K}{1.K} + \frac{z^{K+1}}{1.K.\ K+1} + \ldots ..$$

$$B_K\ (p, q) = \int_{0K}^{1} t^{p-k+1}\ (1-t)^{q-k+1}\ dt, \quad p, q\ \varepsilon\ B_K$$

3. We now prove the following theorem

Theorem :

$$\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} a_n\ b_r = \sum_{n=0}^{\infty} \sum_{r=0}^{[n/\lambda]} a_{n-\lambda r}\ b_r$$

and

$$\sum_{n=0}^{\infty} \sum_{r=0}^{[n/\lambda]} a_n\ b_r = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} a_{n+\lambda r}\ b_r$$

where n and $r\ \varepsilon\ \beta_K$.

Proof :

The product must include all terms of the type $a_i\ d_j\ (i, j\ \varepsilon\ \beta_K)$ for non-negative intergal i and j. Therefore we consider the array :

$$a_0\ b_0 + a_0\ b_1 + \ldots + a_0\ b_{mk} + a_0\ b_{mk+1} + \ldots ..$$

$$a_1\ b_0 + a_1\ b_1 + \ldots + a_1\ b_{mk} + o_1\ b_{mk+1} + \ldots .$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$a_{mk}\, b_0 + a_{mk}\, b_1 + \dots + a_{mk}\, b_{mk} + a_{mk}\, b_{mk+1} + \dots\dots$$

$$a_{mk+1}\, b_0 + a_{mk+1}\, b_1 + \dots + a_{mk+1}\, b_{mk} + a_{mk+1}\, b_{mk} + \dots\dots$$

The product of the series is formed by grouping the above array in finite diagonals according to the following scheme :

Let $\lambda = mk$. Then

$$P_{sk,mk} = a_{sk}\, b_0 + a_{(s-m)k}\, b_1 + a_{(s-2m)k}\, b_K + \dots$$

and

$$P_{sk+1,\, mk} = a_{sk+1}\, b_0 + a_{(s-m)k+1}\, b_1 + a_{(s-2m)k+1} b_K + \dots$$

The number of terms will be finite and depends upon the number

$$\left[\frac{sk}{mk}\right] \text{ and } \left[\frac{sk+1}{mk}\right]$$

In the same way when $\lambda = mk + 1$ we get

$$P_{sk,mk+1} = a_{sk}\, b_0 + a_{(s-m-1)k+1}\, b_1 + a_{(s-2m-1)k+1}\, b_K + \dots\dots$$

and

$$P_{sk+1,mk+1} = a_{sk+1}\, b_0 + a_{(s-m)k}\, b_1 + a_{(s-2m)k}\, b_K + \dots\dots$$

In all these possible cases we see from the tables that

$$P_{sk,mk} = a_{sK-0,mK}\, b_0 + a_{sk\ 1,mK}\, b_1 + a_{sK-K.mK}\, b_K + \dots\dots$$

$$P_{sk+1,mK} = a_{sK+1-0.mK}\, b_0 + a_{sk+1-1,mK}\, b_1 + a_{sK+1-K.mK}\, b_K + \dots\dots$$

$$P_{sK,\, mK+1} = a_{sK-0(mK+1)}\, b_0 + a_{sK-1.\,(mK+1)}\, b_1 + a_{sK-K\ (mK+1)}\, b_K + \dots\dots$$

and

$$P_{sK+1,mK+1} = a_{sK+1-0.(mk+1)}\, b_0 + a_{sK+1-1.\,(mK+1)}\, b_1 + a_{sK+1-K\ (mK+1)}\, b_K + \dots\dots$$

$$i.e.\ P_{n,\lambda} = \sum_{n,r\varepsilon\beta_K}^{[n/\lambda]} a_{n-\lambda r}\, b_r$$

Hence we have proved the theorem. In the same way we can prove the second part of this theorem.

4. Generalized Leibnitz's Theorem.

We define

$$D^n_K(u.\,V) = D^n_K u.\,V + \binom{n}{1} D^{n-1}_K u \;\; D_K V + \binom{n}{2} D^{n-2}_K u.\; D^K_K V + \dots + + u.\, D^n_K V$$

where $n \varepsilon B_K$

We should note that the restriction imposed by the various authors on the operator D_K according to different spaces seems to be unnecessary. For example they define :

$$D^K_K \, z^{mK} = \frac{d}{dz} \frac{1}{zK-2} \frac{d}{dz} \; z^{mK} = mK.\,[\,(m-1)k+1\,].\; z^{(m-1)K}$$

But if we consider the elements of the field B_K in view of the tables we get at once,

$$D^K_K \; z^{mK} = mK.\,[\,(m-1)K+1]\;\; z^{(m-1)K} \text{ since } (m-1)K+1+1=mK$$

Now let us define the operator D_K as

$$\underset{n\to 0}{Lt.} \; \frac{\varphi_K(z,\, h) - \varphi_K(z)}{h} = D_K \;\varphi_K\;(z)$$

We can also prove very easily that

$$D_K\,(u.\,v) = D_K\,u.\;v + D_K\,v.\;u.$$

Now let us operate on both sides of the generalized Leibnitz's theorem by the operator D_k We get

$$\begin{aligned} D^{n+1}_K \;(u.V) &= \sum_{r\varepsilon B_K} D_K \left[\binom{n}{r} \; D^{n-r}_K \; u.\;\; D^r_K \; V \right] \\ &= \sum_{r\varepsilon B_K} \binom{n+1}{r} \;\; D^{n+1-r}_K \;\; u.\;\; D^r_K \; V \end{aligned}$$

Hence we have proved the theorem.

5. Generalized Maclaurins Theorem :

We define it as follows.

$$f_K\;(x) = f_K\,(0) + x\, f^1_K\,(0) + \frac{x^K}{1.K}\;\; f^K_K\,(0) + \frac{x^{K+1}}{1\,K.K+1}\;\; f^{K+1}_K\;\;(0) + \dots$$

Proof : Suppose

$$f_K\;\;(x) = A_0 + A_1\,x + \;A_K\;\;x^K + A_{K+1}\;\;x^{K+1} + \dots$$

Then by successive differentiation, we get,

$$f'_K(x) = A_1 + K A_K x + (K+1) A_{K+1} \; x^K + \dots .$$

$$f_K^K(x) = K.\,1\, A_K + (K+1)\, K\, A_{K+1} \; x + \dots .$$

$$\dots \qquad \dots \qquad \dots \qquad \dots$$

Putting $x = 0$, we find

$$A_0 = f_K(0),\ A_1 = f'_K(0),\ A_K = \frac{f_K^K(0)}{1.\,K}, \dots .$$

hence the theorem follows.

6. We can easily prove the following results :

$$ {}_K F_1\left[\begin{matrix} a,\ b \\ c \end{matrix}\ 1\right] = \frac{\Gamma(c)\ \Gamma(c-a-b)}{\Gamma(c-a)\ \Gamma(c-b)},\ a, b, c\ \varepsilon\ B_K$$

and

$$(1-z)^{-a}\,{}_K F_1\left[\begin{matrix} a,\ c-b \\ c \end{matrix}\ \frac{-z}{1-z}\right] = {}_K F_1\left[\begin{matrix} a,\ b \\ c \end{matrix}\ z\right]\ a, b, c\ \varepsilon\ B_K$$

this has generalized Bailey's series[1]

(A) If

$$(1-x)^{-b}\ {}_r F_s\left[\begin{matrix} a_1, a_2, \dots, a_r \\ \rho_1, \dots, \rho_s \end{matrix}\ c\left(\frac{\lambda x}{1-x}\right)^{\lambda}\right] = \sum_{r\ \varepsilon\ B_K} a_r\, x^r$$

then

$$E_K(x)\ {}_r F_{s+\lambda}\left[\begin{matrix} a_1, a_2, \dots, a_r \\ \rho_1, \dots, \rho_s[\lambda, p] \end{matrix}\ c x^{\lambda}\right] = \sum_{r\ \varepsilon\ B_K} \frac{a_r\, x^r}{(p)_r}$$

where $$[\lambda, p] = \left(\frac{p}{\lambda}\right)\left(\frac{p+1}{\lambda}\right)\left(\frac{p+K}{\lambda}\right)\dots .$$

Also if,

(B)

$$E_K(x)\ {}_r F_{s+\lambda}\left[\begin{matrix} a_1, \dots a_r \\ \rho_1 \dots \rho_s\ [\lambda, p] \end{matrix}\ c x^{\lambda}\right] = \sum_{n\ \varepsilon\ B_K} \frac{a_n\, x^n}{(p)_n}$$

then

$$(1-x)^{-q}\ {}_{r+\lambda} F_{s+\lambda}\left[\begin{matrix} a_1, \dots a_r, [\lambda, q] \\ \rho_1, \dots \rho_s, [\lambda, p] \end{matrix}\ c\left(\frac{\lambda x}{1-x}\right)^{\lambda}\right] = \sum_{n\ \varepsilon\ B_K} \frac{a_r\, (q)_n}{(p)_n}\, x^n$$

With the help of these series we can obtain the following relations as particular cases :

Put $\lambda = s = 1,\ r = K$, in (A), then

$$E_K(x)\ {}_K F_K\left[\begin{matrix} \alpha, \rho-\beta \\ \rho, \alpha-\nu \end{matrix}\ -x\right] = a_0 + \frac{(a)_1\, x}{(\alpha-\nu)_1} + \frac{(a)_K\, x^K}{(a-\nu)_K} + \dots .$$

where

$$(1-x)^{\nu}\ {}_KF_1\left[\begin{matrix}\alpha,\beta\\ \rho\end{matrix}\ x\right] = a_0 + (a)_1\,x + (a)_K\,x^K$$

If $\nu = 0$ we obtain

$${}_1F_1\left[\begin{matrix}\rho-\beta\\ \rho\end{matrix}\ -x\right] = E_K(-x).\quad {}_1F_1\left[\begin{matrix}\beta\\ \rho\end{matrix}\ x\right]$$

Again if $\lambda = K$, $r = K$, $s = 1$, we obtain [in A]

$$E_K(x)\ {}_KF_{K+1}\left[\begin{matrix}\alpha,\ \dfrac{\alpha K+1}{K}\\ \dfrac{\rho K+1}{K},\ \dfrac{p}{K},\ \dfrac{p+1}{K}\end{matrix}\left(\frac{x}{K}\right)^K\right] = b_0 + \frac{b_1\,x}{(p)_1} + \ldots$$

where

$$(1-x)^{-p}\ {}_KF_1\left[\begin{matrix}\alpha,\ \alpha K+1\\ \dfrac{\rho K+1}{K}\end{matrix}\left(\frac{x}{1-x}\right)^K\right] = b_0 + b_1\,x + b_K\,x^K + \ldots.$$

From B, $\lambda = r = 1$, $s = 0$, $c = -1$, then

$$(1-x)^{-q}\ {}_KF_1\left[\begin{matrix}a,\ q\\ p\end{matrix}\ \frac{-x}{1-x}\right] = \sum_{n\,\varepsilon\,B_K}\frac{a_n\,(q)_n}{(p)_n}\,x^n \text{ where}$$

$${}_1F_1\left[\begin{matrix}p-a\\ p\end{matrix}\ x\right] = E_K(x)\ {}_1F_1\left[\begin{matrix}a\\ p\end{matrix}\ -x\right] = \sum_{n\,\varepsilon\,B_K}\frac{a_n}{(p)_n}\,x^n$$

Again if $\lambda = r = 1$, $s = 0$, $c = -K$, then

$$(1-x)^{aK}\ {}_KF_1\left[\begin{matrix}a,\ a+1/K\\ \rho+1/K\end{matrix}\left(\frac{x}{1-x}\right)^K\right] = {}_KF_1\left[\begin{matrix}aK,\ \rho\\ \rho K\end{matrix}\ Kx\right]$$

7. Hermite Polynomials :

We define the generalized Hermite Polynomials, as,

$$E_K\left(Kxt-t^K\right) = \sum_{r\varepsilon B_K}\frac{{}_KH_r(x)\,t^r}{r\,!}$$

hence,

$${}_KH_n(x) = D_K^n\left[E_K\,(Kxt-t^K)\right]_{t=0}\quad n,t\varepsilon B_K$$

$$\text{i.e. } E_K(-x^K)\ {}_KH_n(x) = D_K^n\left[E_K[-(x-t^K)]\right]\Big|_{t=0}$$

$$= (-1)^n\ D_K^n\left[E_K(-x)^K\right],\ n\varepsilon B_K$$

This definition was given by Chattarjee [2]

8. Laguerre Polynomials :

We define as :

$$ {}_{K}L_{n}^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n\,!}\ {}_1F_1\left[\begin{matrix}-n\\ 1+\alpha\end{matrix}\ x\right]\ n\varepsilon B_K $$

We can show very easily that

$$ {}_{K}L_{n}^{(\alpha)}(x) = x^{-\alpha}\, E_K(x)/n\,!\ .\ D_K^n\left[E_K(-x).\,x^{n+\alpha}\right] $$

9. Jacobi Polynomials :

$$ \text{Let } {}_{K}P_{n}^{\alpha,\beta}(x) = \sum_{r,n\varepsilon B_K} \frac{(1+\alpha)_n\,(1+\beta)_n}{(n-r)\,!\,r\,!\,1+\beta)_{n-r}} \left(\frac{x-1}{K}\right)^r \left(\frac{x+1}{K}\right)^{n-r} $$

$$ = \frac{1}{K^n}\cdot\frac{(x-1)^{-\alpha}\,(x+1)^{-\beta}}{n\,!}\sum_{r\varepsilon K}\binom{n}{r} D_K^{n-r}\,(x-1)^{n+\alpha}\ D^r\left((x+1)\right)^{n+\beta} $$

$$ = \frac{(x-1)^{-\alpha}(x+1)^{-\beta}}{K^n\quad n\,!}\,D_K^n\left[(x-1)^{n+\alpha}\ (x+1)^{n+\beta}\right] $$

If $\alpha = \beta = 0$, we obtain

$$ {}_KP_n(x) = \frac{1}{K^n\ n\,!}\ D_K^n\ (x^K-1)^n, \qquad n\varepsilon B_K $$

This result, with slight changes is due to Kharadze [3].

I conclude this paper with the hope that these methods will give more results very soon.

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SOME KERNELS IN THE HANKEL TRANSFORM OF TWO VARIABLES—II

By

R. P. GUPTA

D. S. B. Government College, Nainital

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ABSTRACT

In this paper, the integrals

$$\int_0^\infty \int_0^\infty x^{s+\gamma+n\beta-1}\, y^{\alpha+\beta-1}\, e^{-x}\, {}_1F_1(a\,;\,b\,;\,x)\; I_\nu\left(\tfrac{1}{2}x^n y\right) K_\rho\left(\tfrac{1}{2}{}^n xy\right) dx\, dy$$

and

$$\int_0^\infty \int_0^\infty x^{s+\gamma+n\beta-1}\, y^{\alpha+\beta-1}\, e^{-x}\, {}_1F_1(a\,;\,b\,;\,x)\; J_\rho\left(\sqrt{\tfrac{1}{2}}x^n y\right) K_\rho\left(\sqrt{\tfrac{1}{2}}x^n y\right) dx\, dy$$

have been evaluated by using the Reed's theorem. Twenty kernels transforming $R_{\nu,\rho}$ into $R_{\mu,\lambda}$ have been investigated by giving particular values to the parameters. A self-reciprocal function has also been determined

INTRODUCTION

1. The well-known Hankel transform in one variable, *viz.*

$$g(x) = \int_0^\infty (xy)^{\frac{1}{2}} J_\nu(xy)\, f(y)\, dy$$

has been extended to two variables and is defined by the integral equation

$$g(x, y) = \int_0^\infty \int_0^\infty (x\,\xi\, y\, \eta)^{\frac{1}{2}} J_\rho(\eta y)\, J_\nu(\xi x)\, f(\xi, \eta)\, d\xi\, d\eta$$

Gray [1], Erdelyi [2] and Owen [3] have investigated some self-reciprocal functions involving two complex variables under this transform. R. P. Agarwal [4] has given two formulae connecting different classes of self-reciprocal functions of different orders in two variables. Following the analogy of self reciprocal function of one variable, we define the kernel in two variables in the following manner.

If two functions $f\,(u,u')$ and $g\,(x, y)$ are connected by the relation

$$g\,(x, y) = \int_0^\infty \int_0^\infty P\,(xu, yu')\, f\,(u, u')\; du\; du'\,,$$

where $f\,(u\,u')$ is $R_{\nu,\rho}$ and $g\,(x, y)$ is $R_{\mu,\lambda}$; then $P\,(x, y)$ is called the kernel in two variables for transforming $R_{\nu,\rho}$ into $R_{\mu,\lambda}$ and vice-versa.

Recently I have found some kernels in two variables [5]. The object of this paper is to find more kernels and to employ them to find new self-reciprocal functions.

2. Consider the integral

$$I = \int_0^\infty \int_0^\infty x^{s+\gamma+n\beta-1}\, y^{\alpha+\beta-1}\, e^{-x}\, {}_1F_1(a\,;b\,;x)\, I_\nu\left(\tfrac{1}{2}x^n y\right) K_\rho\left(\tfrac{1}{2}x^n y\right) dx\, dy$$

Evaluating, we get

$$= \frac{2^{\gamma+n\beta+s-a+\beta+\alpha-3}\, \Gamma(b)\, \Gamma(\frac{1}{2}\gamma+\frac{1}{2}n\beta+\frac{1}{2}s)\, \Gamma\frac{1}{2}(1+\gamma+n\beta+s)\, \Gamma\left(\frac{b}{a}-\frac{a}{2}-\frac{\gamma}{2}-\frac{n\beta}{2}-\frac{s}{2}\right)}{\pi\, \Gamma(b-a)\, \Gamma\left(\frac{b}{2}-\frac{\gamma}{2}-\frac{n\beta}{2}-\frac{s}{2}\right) \Gamma\left(\frac{1}{2}+\frac{b}{2}-\frac{\gamma}{2}-\frac{n\beta}{2}-\frac{s}{2}\right)}$$

$$\times \frac{\Gamma\frac{1}{2}(1+b-a-\gamma-n\beta-s)\, \Gamma\frac{1}{2}(1-\beta-\alpha)\, \Gamma\left(1-\frac{\beta}{2}-\frac{\alpha}{2}\right) \Gamma\frac{1}{2}(\nu+\beta+\rho+\alpha)\, \Gamma\frac{1}{2}(\nu+\beta+\alpha-\rho)}{\Gamma\{\frac{1}{2}(\nu-\beta-\alpha+\rho)+1\}\, \Gamma\{\frac{1}{2}(\nu-\beta-\alpha-\rho)+1\}}$$

provided $Re(-\nu \pm \rho) < Re(\alpha+\beta) < 1$, $Re(b-a) > Re(\gamma+n\beta+s) > 0$, $Re\, b > 0$.

Since the integrals involved are absolutely convergent, it follows by the Reed's theorem[6]

$$x^{\gamma+n\beta}\, y^{\beta}\, e^{-x}\, {}_1F_1(a\,;b\,;x)\, I_\nu(\tfrac{1}{2}x^n y)\, K_\rho(\tfrac{1}{2}x^n y)$$

$$= \frac{2^{\gamma+n\beta-a+\beta-3}\, \Gamma(b)}{\pi\, \Gamma(b-a)\, (2\pi i)^2}$$

$$\times \int_{c'-i\infty}^{c'+i\infty} \int_{c-i\infty}^{c+i\infty} 2^{\alpha+s} \frac{\Gamma\left(\frac{\gamma}{2}+\frac{n\beta}{2}+\frac{s}{2}\right) \Gamma\left(\frac{1}{2}+\frac{\gamma}{2}+\frac{n\beta}{2}+\frac{s}{2}\right) \Gamma\left(\frac{b}{2}-\frac{a}{2}-\frac{\gamma}{2}-\frac{n\beta}{2}-\frac{s}{2}\right)}{\Gamma\left(\frac{b}{2}-\frac{\gamma}{2}-\frac{n\beta}{2}-\frac{s}{2}\right) \Gamma\left(\frac{1}{2}+\frac{b}{2}-\frac{\gamma}{2}-\frac{n\beta}{2}-\frac{s}{2}\right)}$$

$$\times \frac{\Gamma(\frac{1}{2}+\frac{b}{2}-\frac{a}{2}-\frac{\gamma}{2}-\frac{n\beta}{2}-\frac{s}{2})\, \Gamma\frac{1}{2}(1-\beta-\alpha)\, \Gamma(1-\frac{\beta}{2}-\frac{\alpha}{2})\, \Gamma\frac{1}{2}(\nu+\beta+\rho+\alpha)\, \Gamma\frac{1}{2}(\nu+\beta-\rho+\alpha)}{\Gamma\{\frac{1}{2}(\nu-\beta-\alpha+\rho)+1\}\, \Gamma[\frac{1}{2}\{\nu-\beta-\alpha-\rho\}+1]}$$

$$\times\, x^{-s}\, y^{-\alpha}\, ds\, d\alpha.$$

Case 1. $\gamma = b-\varepsilon-\frac{1}{2}$, $a = 2\varepsilon-b$, $\beta = 0$, $\nu = -\rho = \frac{1}{2}\mu-\frac{1}{4}$, we get

$$x^{b-\varepsilon-\frac{1}{2}}\, e^{-x}\, {}_1F_1(2\varepsilon-b\,;b\,;x)\, I_{\frac{1}{2}\mu-\frac{1}{4}}(\tfrac{1}{2}x^n y)\, K_{\frac{1}{4}-\frac{1}{2}\mu}(\tfrac{1}{2}x^n y)$$

$$= \frac{1}{(2\pi i)^2}\int_{c'-i\infty}^{c'+i\infty} \int_{c-i\infty}^{c+i\infty} 2^{\alpha+s}\, \Gamma(-\tfrac{1}{4}+\tfrac{1}{2}\varepsilon+\tfrac{1}{2}s)\, \Gamma(\tfrac{1}{4}+\tfrac{1}{2}\varepsilon+\tfrac{1}{2}s)\, \Gamma[\tfrac{1}{4}+\tfrac{1}{2}(\mu-1)+\tfrac{1}{2}\alpha]$$

$$\times\, \Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}\alpha)\, \chi\,(s, \alpha)\, x^{-s}\, y^{-\alpha}\, ds\, d\alpha$$

where

$$X(s, \alpha) = \frac{2^{2b-3\varepsilon-\frac{7}{2}}}{\pi} \frac{\Gamma(b)\ \Gamma(-\frac{1}{4}-\frac{\varepsilon}{2}+\frac{b}{2}+\frac{s}{2})\ \Gamma(\frac{1}{4}+\frac{b}{2}-\frac{\varepsilon}{2}+\frac{s}{2})}{\Gamma(2b-2\varepsilon)\ \Gamma(-\frac{1}{4}+\frac{\varepsilon}{2}+\frac{s}{2})\ \Gamma(\frac{1}{4}+\frac{\varepsilon}{2}+\frac{s}{2})}$$

$$\times \frac{\Gamma(\frac{1}{4}+\frac{b}{2}-\frac{\varepsilon}{2}-\frac{s}{2})\ \Gamma(\frac{3}{4}+\frac{b-\varepsilon-s}{2})\ \Gamma(\frac{\alpha}{2})\ \Gamma(\frac{1}{2}-\frac{\alpha}{2})}{\Gamma(\frac{1}{4}+\frac{\varepsilon}{2}-\frac{s}{2})\quad \Gamma(\frac{3}{4}+\frac{\varepsilon}{2}-\frac{s}{2})}$$

$$= X(1-s,\ 1-\alpha).$$

Hence the function

$$x^{b-\varepsilon-\frac{1}{2}}\ e^{-x}\ {}_1F_1(2\varepsilon-b\ ;\ b\ ;\ x)\ I_{\frac{1}{2}\mu-\frac{1}{4}}(\tfrac{1}{2}x^n y)\ K_{\frac{1}{4}-\frac{1}{2}\mu}(\tfrac{1}{2}x^n y)$$

is a kernel transforming

$$R_{\varepsilon-1,\ \mu-1} \text{ to } R_{\varepsilon,\mu}$$

provided $Re\ \varepsilon < Re\ b-\frac{1}{2}$, $Re\ b > 0$, $Re\ \mu \geqslant \frac{1}{2}$.

Case 2. For $\gamma = \varepsilon-\frac{1}{2}$, $a = 0$, $\beta = 0$, $\nu = -\rho = \frac{1}{2}\mu-\frac{1}{4}$, we get

$$x^{\varepsilon-\frac{1}{2}}\quad e^{-x}\ I_{\frac{1}{2}\mu-\frac{1}{4}}(\tfrac{1}{2}x^n y)\ K_{\frac{1}{4}-\frac{1}{2}\mu}(\tfrac{1}{2}x^n y)$$

as a kernel transforming

$$R_{\varepsilon-1,\mu-1} \text{ to } R_{\varepsilon,\mu}$$

provided $Re\ \varepsilon > \frac{1}{2}$, $Re\ \mu \geqslant \frac{1}{2}$.

Case 3. For $\gamma = \dfrac{\varepsilon}{3}-\frac{1}{2}$, $a = \dfrac{2\varepsilon}{3}$, $b = \dfrac{4\varepsilon}{3}$, $\nu = -\rho = \frac{1}{2}\mu-\frac{1}{4}$, $\beta = 0$, we get

$$x^{-\frac{\varepsilon}{3}}\ e^{-\frac{1}{2}x}\ I_{\frac{2\varepsilon}{3}-\frac{1}{2}}(\tfrac{1}{2}x)\ I_{\frac{1}{2}\mu-\frac{1}{4}}(\tfrac{1}{2}x^n y)\ K_{\frac{1}{4}-\frac{1}{2}\mu}(\tfrac{1}{2}x^n y)$$

as a kernel transforming

$$R_{\varepsilon-1,\mu-1} \text{ to } R_{\varepsilon,\mu}$$

provided $Re\ \varepsilon > 3/2$, $Re\ \mu \geqslant \frac{1}{2}$.

Case 4. For $\gamma = \varepsilon-\frac{1}{2}$, $a = -1$, $b = 2\varepsilon-2$, $\beta = 0$, $\nu = -\rho = \frac{1}{2}\mu-\frac{1}{4}$, we get

$$x^{\varepsilon-\frac{1}{2}}\ e^{-x}\ {}_1F_1(-1\ ;\ 2\varepsilon-2\ ;\ x)\ I_{\frac{1}{2}\mu-\frac{1}{4}}(\tfrac{1}{2}x^n y)\ K_{\frac{1}{4}-\frac{1}{2}\mu}(\tfrac{1}{2}x^n y)$$

as a kernel transforming

$$R_{\varepsilon-3,\mu-1} \text{ to } R_{\varepsilon,\mu},$$

provided $Re\ \varepsilon > 2$, $Re\ \mu \geqslant \frac{1}{2}$.

Case 5. For $\gamma = \varepsilon + \frac{1}{2}$, $a = -1$, $b = 2\varepsilon + 1$, $\beta = 0$, $\nu = -\rho = \frac{1}{2}\mu - \frac{1}{4}$, we get

$$x^{\varepsilon + \frac{1}{2}}\, e^{-x}\, {}_1F_1(-1\,;\, 2\varepsilon + 1\,;\, x)\, I_{\frac{1}{2}\mu - \frac{1}{4}}(\tfrac{1}{2}x^n y)\, K_{\frac{1}{4} - \frac{1}{2}\mu}(\tfrac{1}{2}x^n y)$$

as a kernel transforming

$$R_{\varepsilon-1,\mu-1} \text{ to } R_{\varepsilon,\mu},$$

provided $Re\ \varepsilon > 0$, $Re\ \mu \geqslant \frac{1}{2}$.

Case 6. For $\gamma = b - \frac{n}{6} + \frac{n\mu}{3} - \varepsilon - \frac{1}{2}$, $a = 2\varepsilon - b$, $\beta = \frac{1}{6} - \frac{\mu}{3}$, $\nu = \rho = \frac{\mu}{3} - \frac{1}{6}$, we get

$$x^{b-\varepsilon-\frac{1}{2}}\, y^{\frac{1}{6}-\frac{\mu}{3}}\, e^{-x}\, {}_1F_1(2\varepsilon - b\,;\, b\,;\, x)\, I_{\frac{\mu}{3}-\frac{1}{6}}(\tfrac{1}{2}x^n y)\, K_{\frac{\mu}{3}-\frac{1}{6}}(\tfrac{1}{2}x^n y)$$

as a kernel transforming

$$R_{\varepsilon-1,-\frac{\mu+1}{3}} \text{ to } R_{\varepsilon,\mu},$$

provided $Re\ \varepsilon < Re\ b - \frac{1}{2}$, $Re\ b > 0$, $\frac{1}{2} \geqslant Re\ \mu \geqslant -\frac{1}{2}$.

Case 7. For $\gamma = \varepsilon - \frac{n}{6} + \frac{n\mu}{3} - \frac{1}{2}$, $a = 0$, $\beta = \frac{1}{6} - \frac{\mu}{3}$, $\nu = \rho = \frac{\mu}{3} - \frac{1}{6}$, we get

$$x^{\varepsilon-\frac{1}{2}}\, y^{\frac{1}{6}-\frac{\mu}{3}}\, e^{-x}\, I_{\frac{\mu}{3}-\frac{1}{6}}(\tfrac{1}{2}x^n y)\, K_{\frac{\mu}{3}-\frac{1}{6}}(\tfrac{1}{2}x^n y)$$

as a kernel transforming

$$R_{\varepsilon-1,-\frac{\mu+1}{3}} \text{ to } R_{\varepsilon,\mu},$$

provided $Re\ \varepsilon > \frac{1}{2}$, $\frac{1}{2} \geqslant Re\ \mu \geqslant -\frac{1}{2}$.

Case 8. For $\gamma = \frac{\varepsilon}{3} - \frac{n}{6} + \frac{n\mu}{3} - \frac{1}{2}$, $\beta = \frac{1}{6} - \frac{\mu}{3}$, $a = \frac{2\varepsilon}{3}$, $b = \frac{4\varepsilon}{3}$, $\nu = \rho = \frac{\mu}{3} - \frac{1}{6}$, we get

$$x^{-\frac{\varepsilon}{3}}\, y^{\frac{1}{6}-\frac{\mu}{3}}\, e^{-\frac{1}{2}x}\, I_{\frac{2\varepsilon}{3}-\frac{1}{2}}(\tfrac{1}{2}x)\, I_{\frac{\mu}{3}-\frac{1}{6}}(\tfrac{1}{2}x^n y)\, K_{\frac{\mu}{3}-\frac{1}{6}}(\tfrac{1}{2}x^n y)$$

as a kernel transforming

$$R_{\varepsilon-1,-\frac{\mu+1}{3}} \text{ to } R_{\varepsilon,\mu},$$

provided $Re\ \varepsilon > \frac{3}{2}$, $\frac{1}{2} \geqslant Re\ \mu \geqslant -\frac{1}{2}$.

Case 9. For $\gamma = \varepsilon - \frac{n}{6} + \frac{n\mu}{3} - \frac{1}{2}$, $a = -1$, $b = 2\varepsilon - 2$, $\beta = \frac{1}{6} - \frac{\mu}{3}$, $\nu = \rho = \frac{\mu}{3} - \frac{1}{6}$, we get

$$x^{\varepsilon-\frac{1}{2}}\, y^{\frac{1}{6}-\frac{\mu}{3}}\, e^{-x}\ {}_1F_1(-1\,;\,2\varepsilon-2\,;\,x)\ I_{\frac{\mu}{3}-\frac{1}{6}}(\tfrac{1}{2}x^n y)\ K_{\frac{\mu}{3}-\frac{1}{6}}(\tfrac{1}{2}x^n y)$$

as a kernel transforming

$$R_{\varepsilon-3,-\frac{\mu+1}{3}} \text{ to } R_{\varepsilon,\mu},$$

provided $Re\ \varepsilon > 2$, $\frac{1}{2} \geqslant Re\ \mu \geqslant -\frac{1}{2}$.

Case 10. For $\gamma = \varepsilon - \frac{n}{6} + \frac{n\mu}{3} + \frac{1}{2}$, $a = -1$, $b = 2\varepsilon + 1$, $\beta = \frac{1}{6} - \frac{\mu}{3}$, $\nu = \rho = \frac{\mu}{3} - \frac{1}{6}$, we get

$$x^{\varepsilon+\frac{1}{2}}\, y^{\frac{1}{6}-\frac{\mu}{3}}\, e^{-x}\ {}_1F_1(-1\,;\,2\varepsilon+1\,;\,x)\ I_{\frac{\mu}{3}-\frac{1}{6}}(\tfrac{1}{2}x^n y)\ K_{\frac{\mu}{3}-\frac{1}{6}}(\tfrac{1}{2}x^n y)$$

as a kernel transforming

$$R_{\varepsilon-1,-\frac{\mu+1}{3}} \text{ to } R_{\varepsilon,\mu},$$

provided $Re\ \varepsilon > 0$, $\frac{1}{2} \geqslant Re\ \mu \geqslant -\frac{1}{2}$.

3. Consider the integral

$$I = \int_0^\infty \int_0^\infty x^{s+\gamma+n\beta-1}\, y^{\alpha+\beta-1}\, e^{-x}\ {}_1F_1(a\,;\,b\,;\,x)\, J_\rho(\sqrt{\tfrac{1}{2}}x^n y)\, K_\rho(\sqrt{\tfrac{1}{2}}x^n y)\, dx\, dy$$

$$= \frac{2^{\gamma+n\beta-a+\beta-3+s+\alpha}\ \Gamma(b)\ \Gamma(\frac{\gamma}{2}+\frac{n\beta}{2}+\frac{s}{2})\ \Gamma\frac{1}{2}(1+\gamma+n\beta+s)\ \Gamma\frac{1}{2}(b-a-\gamma-n\beta-s)}{\pi\ \Gamma(b-a)\ \Gamma\frac{1}{2}(b-\gamma-n\beta-s)\ \Gamma\frac{1}{2}(1+b-\gamma-n\beta-s)}$$

$$\times \frac{\Gamma\frac{1}{2}(1+b-a-\gamma-n\beta-s)\ \Gamma(\rho+\frac{\beta}{2}+\frac{\alpha}{2})\ \Gamma\frac{1}{2}(\alpha+\beta)}{\Gamma(\frac{\rho}{2}+\frac{\beta}{2}+\frac{\alpha}{4}+\frac{1}{2})\ \Gamma(\frac{\rho}{2}-\frac{\alpha}{4}-\frac{\beta}{4}+1)}$$

provided $Re(b-a) > Re(\gamma+n\beta+s) > 0$, $Re\ b > 0$, $Re(\alpha+\beta+\rho) > |\ Re\ \rho\ |$.

Since the integral involved are absolutely convergent, it follows by the Reed's theorem

$$x^{\gamma+n\beta}\, y^{\beta}\, e^{-x}\ {}_1F_1(a\,;\,b\,;\,x)\, J_\rho(\sqrt{\tfrac{1}{2}}x^n y)\, K_\rho(\sqrt{\tfrac{1}{2}}x^n y)$$

$$= \frac{2^{\gamma+n\beta-a+\beta-3}\ \Gamma(b)}{\pi\ \Gamma(b-a)\ (2\pi i)^2} \int_{c'-i\infty}^{c'+i\infty} \int_{c-i\infty}^{c+i\infty} 2^{\alpha+s}\, \frac{\Gamma\frac{1}{2}(\gamma+n\beta+s)\ \Gamma\frac{1}{2}(1+\gamma+n\beta+s)}{\Gamma\frac{1}{2}(b-\gamma-n\beta-s)}$$

$$\times \frac{\Gamma\frac{1}{2}(b-a-\gamma-n\beta-s)\ \Gamma\frac{1}{2}(1+b-a-\gamma-n\beta-s)\ \Gamma\frac{1}{2}(\alpha+\beta+2\rho)\ \Gamma\frac{1}{2}(\alpha+\beta)}{\Gamma\frac{1}{2}(1+b-\gamma-n\beta-s)\ \Gamma\frac{1}{4}(2\rho+\beta+\alpha+2)\ \Gamma\frac{1}{4}(2\rho-\alpha-\beta+2)}\, x^{-s}\, y^{-\alpha}\, ds\, d\alpha$$

Case 1. If $\gamma = b - \varepsilon - \frac{1}{2}n - \frac{1}{2}$, $a = 2\varepsilon - b$, $\beta = \frac{1}{2}$, we get

$$x^{b-\varepsilon-\frac{1}{2}}\, y^{\frac{1}{2}}\; e^{-x}\, {}_1F_1(2\varepsilon-b\,;\, b\,;\, x)\, J_\rho(\sqrt{\tfrac{1}{2}}x^n y)\, K_\rho(\sqrt{\tfrac{1}{2}}x^n y)$$

$$= \frac{1}{(2\pi i)^2}\int_{c'-i\infty}^{c'+i\infty}\int_{c-i\infty}^{c+i\infty} 2^{\alpha+s}\,\Gamma(-\tfrac{1}{4}+\tfrac{1}{2}\varepsilon+\tfrac{1}{2}s)\;\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\varepsilon+\tfrac{1}{2}s)\;\Gamma(\tfrac{1}{4}+\rho+\tfrac{1}{2}\alpha)$$

$$\times\; \Gamma(\tfrac{1}{4}+\tfrac{1}{2}\alpha)\; \chi(s,\, \alpha)\; x^{-s}\, y^{-\alpha}\, ds\, d\alpha$$

where

$$\chi(s,\alpha) = \frac{2^{2b-3\varepsilon-\frac{7}{2}-\frac{n}{2}}\;\Gamma(b)\;\Gamma(-\frac{1}{4}-\frac{\varepsilon}{2}+\frac{b}{2}+\frac{s}{2})\quad \Gamma(\frac{1}{4}+\frac{b}{2}-\frac{\varepsilon}{2}+\frac{s}{2})\;\Gamma(\frac{1}{4}+\frac{b}{2}-\frac{\varepsilon}{2}-\frac{s}{2})}{\pi\;\Gamma(2b-2\varepsilon)\;\Gamma(-\frac{1}{4}+\frac{\varepsilon}{2}+\frac{s}{2})\;\;\Gamma(\frac{1}{4}+\frac{\varepsilon}{2}+\frac{s}{2})\;\Gamma(\frac{1}{4}+\frac{\varepsilon}{2}-\frac{s}{2})}$$

$$\times\; \frac{\Gamma(\frac{3}{4}+\frac{b}{2}-\frac{\varepsilon}{2}-\frac{s}{2})}{\Gamma(\frac{3}{4}+\frac{\varepsilon}{2}-\frac{s}{2})\;\Gamma(\frac{5}{8}+\frac{\rho}{2}+\frac{\alpha}{4})\;\Gamma(\frac{7}{8}+\frac{\rho}{2}-\frac{\alpha}{4})}$$

$$= \chi(1-s,\; 1-\alpha)$$

Hence the function

$$x^{b-\varepsilon-\frac{1}{2}}\, y^{\frac{1}{2}}\;\; e^{-x}\, {}_1F_1(2\varepsilon-b\,;\, b\,;\; x)\; J_\rho(\sqrt{\tfrac{1}{2}}\,x^n y)\; K_\rho(\sqrt{\tfrac{1}{2}}\,x^n y)$$

is a kernel transforming

$$R_{\varepsilon-1,2\rho} \text{ to } R_{\varepsilon,0},$$

provided $Re\ \varepsilon < Re\ b - \frac{1}{2}$, $Re\ b > 0$, $Re\ \rho > -\frac{1}{2}$.

Case 2. For $\gamma = \varepsilon - \frac{1}{2}n - \frac{1}{2}$, $a = 0$, $\beta = \frac{1}{2}$, we get

$$x^{\varepsilon-\frac{1}{2}}\, y^{\frac{1}{2}}\; e^{-x}\;\; J_\rho(\sqrt{\tfrac{1}{2}}\,x^n y)\; K_\rho(\sqrt{\tfrac{1}{2}}\,x^n y)$$

as a kernel transforming

$$R_{\varepsilon-1,2\rho} \text{ to } R_{\varepsilon,0},$$

provided $Re\ \varepsilon > \frac{1}{2}$, $Re\ \rho > -\frac{1}{2}$.

Case 3. For $\gamma = \frac{\varepsilon}{3} - \frac{1}{2}n - \frac{1}{2}$, $a = \frac{2\varepsilon}{3}$, $b = \frac{4\varepsilon}{3}$, $\beta = \frac{1}{2}$, we get

$$x^{\frac{\varepsilon}{3}}\, y^{\frac{1}{2}}\, e^{-\frac{1}{2}x}\;\; I_{\frac{2\varepsilon}{3}-\frac{1}{2}}(\tfrac{1}{2}x)\; J_\rho\,(\sqrt{\tfrac{1}{2}}\,x^n y)\;\; K_\rho\,(\sqrt{\tfrac{1}{2}}\,x^n y)$$

as a kernel transforming

$$R_{\varepsilon-1,2\rho} \text{ to } R_{\varepsilon,0},$$

provided $Re\ \varepsilon > \frac{3}{2}$, $Re\ \rho > -\frac{1}{2}$.

Case 4. For $\gamma = \varepsilon - \frac{1}{2}n - \frac{1}{2}$, $a = -1$, $b = 2\varepsilon - 2$, $\beta = \frac{1}{2}$, we get

$$x^{\varepsilon-\frac{1}{2}}\ y^{\frac{1}{2}}\ e^{-x}\ {}_1F_1(-1\ ;\ 2\varepsilon-2\ ;\ x)\ J_\rho(\sqrt{\tfrac{1}{2}}x^ny)\ \ K_\rho(\sqrt{\tfrac{1}{2}}x^ny)$$

as a kernel transforming

$$R_{\varepsilon-3,2\rho} \text{ to } R_{\varepsilon,0},$$

provided $Re\ \varepsilon > 2$, $Re\ \rho > -\frac{1}{2}$.

Case 5. For $\gamma = \varepsilon - \frac{1}{2}n + \frac{1}{2}$, $a = -1$, $b = 2\varepsilon + 1$, $\beta = \frac{1}{2}$, we get

$$x^{\varepsilon+\frac{1}{2}}\ y^{\frac{1}{2}}\ e^{-x}\ {}_1F_1(-1\ ;\ 2\varepsilon+1\ ;\ x)\ J_\rho(\sqrt{\tfrac{1}{2}}x^ny)\ K_\rho(\sqrt{\tfrac{1}{2}}x^ny)$$

as a kernel transforming

$$R_{\varepsilon-1,2\rho} \text{ to } R_{\varepsilon,0},$$

provided $Re\ \varepsilon > 2$, $Re\ \rho > -\frac{1}{2}$.

Case 6. For $\gamma = b - \varepsilon - (5/2)n - \frac{1}{2}$, $a = 2\varepsilon - b$, $\beta = 5/2$, we get

$$x^{b-\varepsilon-\frac{1}{2}}\ y^{5/2}\ e^{-x}\ {}_1F_1(2\varepsilon - b\ ;\ b\ ;\ x)\ J_\rho(\sqrt{\tfrac{1}{2}}x^ny)\ K_\rho(\sqrt{\tfrac{1}{2}}x^ny)$$

as a kernel transforming

$$R_{\varepsilon-1,2\rho} \text{ to } R_{\varepsilon,2},$$

provided $Re\ \varepsilon < Re\ b - \frac{1}{2}$, $Re\ b > 0$, $Re\ \rho > \frac{1}{2}$.

Case 7. For $\gamma = \varepsilon - (5/2)n - \frac{1}{2}$, $a = 0$, $\beta = 5/2$, we get

$$x^{\varepsilon-\frac{1}{2}}\ \ y^{5/2}\ \ e^{-x}\ \ J_\rho(\sqrt{\tfrac{1}{2}}x^ny)\ \ K_\rho(\sqrt{\tfrac{1}{2}}x^ny)$$

as a kernel transforming

$$R_{\varepsilon-1,2\rho} \text{ to } R_{\varepsilon,2}.$$

provided $Re\ \varepsilon > \frac{1}{2}$, $Re\ \rho > \frac{1}{2}$.

Case 8. For $\gamma = \frac{\varepsilon}{3} - (5/2)n - \frac{1}{2}$, $a = \frac{2\varepsilon}{3}$, $b = \frac{4\varepsilon}{3}$, $\beta = 5/2$, we get

$$x^{\frac{-\varepsilon}{3}}\ y^{5/2}\ I_{\frac{2\varepsilon}{3}-\frac{1}{2}}(\tfrac{1}{2}x)\ J_\rho\ (\sqrt{\tfrac{1}{2}}\ x^ny)\ K_\rho\ (\sqrt{\tfrac{1}{2}}\ x^ny)$$

as a kernel transforming

$$R_{\varepsilon-1,2\rho} \text{ to } R_{\varepsilon,2}$$

provided $Re\ \varepsilon > 3/2$, $Re\ \rho > \frac{1}{2}$.

Case 9. For $\gamma = \varepsilon - (5/2)n \cdot \frac{1}{2}$, $a = -1$, $b = 2\varepsilon - 2$, $\beta = 5/2$, we get

$$x^{\varepsilon-\frac{1}{2}}\, y^{5/2}\, e^{-x}\, {}_1F_1(-1\,;\,2\varepsilon-2\,;\,x)\, J_\rho(\sqrt{\tfrac{1}{2}}\, x^n y)\, K_\rho(\sqrt{\tfrac{1}{2}}\, x^n y)$$

as a kernel transforming

$$R_{\varepsilon-3,2\rho} \text{ to } R_{\varepsilon,2}$$

provided $Re\ \varepsilon > 2$, $Re\ \rho > \frac{1}{2}$.

Case 10. For $\gamma = \varepsilon - (5/2)n + \frac{1}{2}$, $a = -1$, $b = 2\varepsilon + 1$, $\beta = 5/2$, we get

$$x^{\varepsilon+\frac{1}{2}}\, y^{5/2}\, e^{-x}\, {}_1F_1(-1\,;\,2\varepsilon+1\,;\,x)\, J_\rho(\sqrt{\tfrac{1}{2}}x^n y)\, K_\rho(\sqrt{\tfrac{1}{2}}x^n y)$$

as a kernel transforming

$$R_{\varepsilon-1,2\rho} \text{ to } R_{\varepsilon,2}$$

provided $Re\ \varepsilon > 0$, $Re\ \rho > \frac{1}{2}$.

Example. We know that

$$(xy)^{\frac{1}{2}(\nu+\rho+1)}\, K_{\frac{1}{2}(\rho-\nu)}(xy), \quad Re\ \nu > -1,\ Re\ \rho > -1$$

is $R_{\nu,\rho}$.

Hence considering the case 9 of § 2,

$$G(x,y) = \int_0^\infty \int_0^\infty (xu)^{\nu-\frac{1}{2}}\, e^{-xu}\, {}_1F_1(-1\,;\,2\nu-2\,;\,xu)\, (yu')^{\frac{1}{6}-\frac{\rho}{3}}\, I_{\frac{\rho}{3}-\frac{1}{6}}(\tfrac{1}{2}x^n y/yu')$$

$$K_{\frac{\rho}{3}-\frac{1}{6}}(\tfrac{1}{2}x^n yu')\, (uu')^{\frac{1}{2}(\nu+\rho+1)}\, K_{\frac{1}{2}(\rho-\nu)}(uu')\, du\, du'$$

$$= \frac{2^{2\nu+\frac{\rho^2}{3}-\frac{5}{6}}\,\Gamma(2\nu-2)}{\Gamma(2\nu-1)\pi}\, x^{-\left[\, n\left(\frac{\nu}{2}+\frac{\rho}{6}+\frac{7}{6}\right)+\frac{1+\rho}{2}+1\,\right]}\, y^{-\left(\frac{\nu}{2}+\frac{\rho}{2}+1\right)}$$

$$\times\, G^{4,4}_{4,6}\left(\frac{x^{2n+2}\, y^2}{4}\,\middle|\,\begin{matrix}\frac{13}{12}+\frac{\nu}{4}+\frac{\rho}{12},\ \frac{3}{4}+\frac{\rho}{4}-\frac{\nu}{4},\ \frac{7}{4}-\frac{\nu}{4}+\frac{\rho}{4},\ \frac{5}{4}-\frac{\nu}{4}+\frac{\rho}{4}\\ \frac{5}{12}+\frac{\nu}{4}+\frac{5\rho}{12},\ -\frac{1}{4}+\frac{\nu}{4}+\frac{\rho}{12},\ \frac{3}{4}+\frac{3\nu}{4}+\frac{\rho}{4},\ \frac{1}{4}+\frac{3\nu}{4}+\frac{\rho}{4},\ \frac{9}{4}-\frac{\nu}{4}+\frac{\rho}{4},\ \frac{7}{4}+\frac{\rho-\nu}{4}\end{matrix}\right)$$

provided $Re\,(\frac{1}{2}\nu + \frac{5\rho}{6}) + \frac{4}{3} > |\,Re(\rho-\nu)\,|$, $Re(\frac{1}{2}\nu + \frac{\rho}{6}) + \frac{5}{3} > |\,Re(\rho-\nu)\,|$.

$$Re\left[\frac{3\nu}{2}+\tfrac{1}{2}\rho-\tfrac{1}{2}+2\min\left(\tfrac{1}{4}+\tfrac{1}{4}\rho-\tfrac{1}{4}\nu,-\tfrac{1}{12}-\tfrac{1}{4}\nu-\frac{\rho}{12},\tfrac{1}{4}-\tfrac{1}{4}\rho+\tfrac{1}{4}\nu\right)\right]>-1$$

and $$2Re\left(\frac{7}{12}-\tfrac{1}{4}\nu-\frac{5\rho}{12},\tfrac{5}{4}-\tfrac{1}{4}\nu-\frac{\rho}{12}\right)+\tfrac{3}{4}-\tfrac{1}{2}\nu+\tfrac{1}{2}\rho<2$$

on using the results [7, pp. 437, 153, 430, 442] and [8].

Thus the function

$$x^{-\left[n\left(\frac{\nu}{2}+\frac{\rho}{6}+\frac{7}{6}\right)+\frac{\nu+\rho}{2}+1\right]}\,y^{-\left(\frac{\nu}{2}+\frac{\rho}{2}+1\right)}$$

$$\times\, G^{4,4}_{4,6}\left(\frac{x^{2n+2}\,y^2}{4}\,\middle|\,\begin{matrix}\frac{13}{12}+\frac{\nu}{4}+\frac{\rho}{12},\frac{3}{4}+\frac{\rho-\nu}{4},\frac{7}{4}+\frac{\rho-\nu}{4},\frac{5}{4}+\frac{\rho-\nu}{4}\\ \frac{5}{12}+\frac{\nu}{4}+\frac{5\rho}{12},-\frac{1}{4}+\frac{\nu}{4}+\frac{\rho}{12},\frac{3+3\nu+\rho}{4},\frac{1+3\nu+\rho}{4},\frac{9-\nu+\rho}{4},\frac{7-\nu+\rho}{4}\end{matrix}\right)$$

is $R_{\nu-3,-\frac{\rho+1}{3}}$.

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INORGANIC CO-ORDINATION COMPLEXES OF BIVALENT NICKEL PART I. CO-ORDINATION COMPLEXES OF NICKEL (II) BENZOATE WITH PRIMARY ALIPHATIC AMINES

By

GOPAL NARAIN *and* P. R. SHUKLA (Miss)

Department of Chemistry, Lucknow University, Lucknow

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ABSTRACT

Nickel (II) benzoate was found to undergo complex formation with primary aliphatic amines and diamines. All the complexes were found to be amorphous coloured solids. The general molecular formula on the basis of analysis and other experimental results turn out to be [Ni $(Am)_4$ $(Ben)_2$] and [Ni $(DAm)_2$ $(Ben)_2$]. Molar conductance in formamide shows the complexes to be non electrolytes and molecular weight measurements confirm this. The general properties of the complexes have been discussed.

During the past many years considerable amount of work has been done on the reaction of amines with inorganic salts of nickel (II), but practically very little work has been carried on the formation of amino-complexes of nickel (II) with organic anions[1-6]. There are also a number of evidences which suggest that the nature of the anion is a contributing factor to the stability[7], coordination number[8] and colour of the complexes[9].

The present investigation has been carried out with a view to study the formation and properties of the compounds of nickel (II) benzoate with primary aliphatic amines and diamines. The benzoate ion besides neutralizing the charge of the metal, can also be co-ordinated to it through the oxygen and it will also be an interesting point to see its effect on the properties of the complexes.

EXPERIMENTAL SECTION

General method of preparation of compound.—About one gm of nickel (II) benzoate was suspended in 15 ml of benzene. A little more than the calculated quantity of amine was added. The reaction mixture was shaked for few hours, filtered, washed to remove excess of amine and dried in a vacuum desiccator first over fused calcium chloride then over phosphorous (V) oxide and analysed.

The ammonia complex was prepared in acetone medium. The complex was dissolved by the addition of an excess of ammonia and then it was crystallised by evaporating the excess ammonia and acetone in a vacuum desiccator.

Analysis.—Metal was estimated as dimethylglyoxime complex, carbon and nitrogen micro-analytically. The amine content was determined as outlined by the author.[10]

(1) *Dibenzoato-tetra (amine) nickel (II)*

The complex is blue in colour, crystalline and soluble in formamide. Found Ni=15·68%, C=44·88%, N=15·08%; $C_{14}H_{22}O_4N_4.Ni$ requires Ni=15·92%, C=45·57%, N=15·19%.

(2) *Dibenzoato-tetra (methylamine) nickel (II)*

The complex is blue in colour, amorphous solid and soluble in formamide. Found Ni=13·82%, C=50·01%, N=12·99%; $C_{18}H_{30}O_4N_4$.Ni requires Ni=13·82%, C=50·86%, N=13·19%.

(3) *Dibenzoato-tetra (ethylamine) nickel (II)*

The complex is blue in colour, amorphous solid and soluble in formamide. Found Ni=12·02%, C=54·06%, N=11 22%; $C_{22}H_{38}O_4N_4$.Ni requires Ni=12·21%, C=54·92%, N=11·65%.

(4) *Dibenzoato-tetra (n-propylamine) nickel (II)*

The complex is blue in colour, amorphous solid and soluble in formamide and nitrobenzene. Found Ni=10 66%, C=57·67%, N=10·22%, $C_{26}H_{46}O_4N_4$.Ni requires Ni=10·93%, C=58·13%, N=10·43%.

(5) *Dibenzoato-tetra (n-butylamine) nickel (II)*

The complex is light blue in colour, amorphous solid and soluble tn formamide. Found Ni=9·73%, C=59·8%, N=9·21%; $C_{30}H_{54}O_4N_4$.Ni requires Ni=9·9%, C=60·73%, N=9·45%.

(6) *Dibenzoato-bis (ethylenediamine) nickel (II)*

The complex is pink in colour, crystalline and soluble in water, alcohol and formamide Found Ni=13·62%, C=50·02%, N=13·21%; $C_{18}H_{26}O_4N_4$.Ni requires Ni=13·95%. C=51·37%, N=13·31%.

(7) *Dibenzoato-bis (propylenediamine) nickel (II)*

The complex is pink in colour, crystalline and soluble in water, alcohol and formamide. Found Ni=13·23%, C=51·5%, N=12·28%; $C_{20}H_{30}O_4N_4$.Ni requires Ni=13·08%. C=53·49%, N=12·48%.

Determination of molar conductance and molecular weight

The conductance measurements were done in formamide at a concentration of 10^{-3}M. The cell used had a cell constant of 0·0245. The molecular weight measurements were also done in formamide. A modified apparatus was used as outlined by the author[11] to avoid supercooling.

TABLE

Results of molar conductance, percentage of total amine and molecular weight

No.	Formulae	Molar Conductance	Total amine Calc.	Total amine Obs.	Molecular Weight Calc.	Molecular Weight Obs.
1.	$[Ni(NH_3)_4(C_6H_5COO)_2]$	0·83 mhos	18 45%	18·33%	368·7	340±10
2.	$[Ni(CH_3NH_2)_4(C_6H_5COO)_2]$	0·56 mhos	29·19%	28·69%	424·7	388±15
3.	$[Ni(C_2H_5NH_2)_4(C_6H_5COO)_2]$	0·93 mhos	37·45%	38·10%	480·7	451±15
4.	$[Ni(C_3H_7NH_2)_4(C_6H_5COO)_2]$	0·36 mhos	43 96%	43 36%	536·7	500±20
5.	$[Ni(C_4H_9NH_2)_4(C_6H_5COO)_2]$	0·88 mhos	49·26%	48·88%	592·7	550±25
6.	$[Ni(en)_2(C_6H_5COO)_2]$	0·55 mhos	28·53%	28·22%	420·7	410±15
7.	$[Ni(pn)_2(C_6H_5COO)_2]$	0·98 mhos	32 98%	32·66%	448·7	430±10

In the above table en = $H_2N\ CH_2CH_2NH_2$

pn = $H_2N\ CH_2CH_2CH_2\ NH_2$

DISCUSSION

On the basis of percentages of constituent elements the general molecular formalae turn out to be $Ni(Ben)_2.4$ Am and $Ni(Ben)_2.2$ DAm, where Ben, Am and DAm represent the benzoate ion, primary aliphatic amines and aliphatic diamines, respectively. These formulae show that four molecules of primary amine and two molecules of diamine are added per molecule of nickel (II) benzoate, which is also confirmed by the estimations of the percentage of total amine content. These complexes are blue and pink in colours, very stable, amorphous and crystalline solids and mostly soluble in formamide. The diamine complexes are also soluble in water. They are neither hygroscopic nor decompose when they come into contact with moisture.

The compounds dissolve in formamide to give very dilute solutions. Measurements of molar conductance in these solutions gave a value less than 1. These values indicate the compounds to be non-electrolytes.

The freezing point determination in formamide gave the normal absolute value of molecular weight. These results suggest that benzoate ions are inside the co-ordination sphere along with the amine. The formulae must be written as $[Ni(Am)_4 (Ben)_2]^0$ and $[Ni(DAm)_2 (Ben)_2]^0$

The structures of these compounds is under investigation. Infra-red, magnetic susceptibility and visible spectrophotometric measurements are being conducted and it is hoped these results will help in elucidating the structure of these compounds.

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COMPARATIVE PERFORMANCE OF THE DRY COMBUSTION AND THE RAPID DICHROMATE METHODS FOR THE DETERMINATION OF ORGANIC CARBON IN DIFFERENT SOIL GROUPS OF INDIA

By

G. C. SHUKLA *and* K. S. TYAGI

Division of Soil Science and Agricultural Chemistry,

Indian Agricultural Research Institute, New Delhi

[Received on 24th March, 1966]

ABSTRACT

Walkley-Black method is extensively used for the estimation of organic carbon in soils. The recovery of total organic carbon, reported by Walkley is 77%. This was verified for different soil groups of India using dry combustion method as standard. The mean recovery values obtained differ considerably from the original.

A large number of promising methods for the estimation of organic carbon in soils are known (Robinson and Williams, 1929 ; Tiurin, 1931, 1934 ; Walkley and Black, 1934, 1935 ; Van Slyke and Folch, 1940 ; Tinsley, 1950). Out of all these methods the Walkley-Black method is the most popular and widely adopted because of its simplicity, rapidity and as it gives better index of the soil fertility. However, it suffers from the disadvantage that it does not estimate highly condensed forms of organic carbon *e.g.* charcoal, coal, graphite etc. Therefore, a suitable recovery factor is used for the determination of total organic carbon. Walkley (1947) found a recovery of 77%. Browning (1938) compared this method with the dry combustion method as recommended in A. O A. C. and reported a mean recovery of 84·7%. He applied the method to 150 samples of different origin and arrived at variations from 92·9 to 110·1%. Tinsley (1950) obtained a recovery of 83·6 $\pm$ 1·3% ; Smith and Weldon (1941) 74%.

An approach towards finding out recovery factor for Indian soils was made by Dewan *et al.* (1959). However, no definite factor for use emerged from this study as differences in soils were not recognised on the basis of natural groups. All the soils were grouped into two classes depending on the presence or absence of carbonates. The composition of the soil as well as the nature of organic matter is related to geoclimatic conditions. Hence, the present investigations were made with due consideration of the different soil groups of India.

EXPERIMENTAL

A variety of soils, surface (0–9″) and sub-surface (9–18″), representing different soil groups found in India, *e.g.* black, red, alluvial, laterite and hilly and forest soils were collected for the present investigations. The soil samples were air dried and crushed to pass 2mm sieve. For the organic carbon determinations the samples were screend through a 0·5 mm sieve. Organic carbon was determined by dry combustion method (Read and Ridgell, 1922) and also by rapid titration method (Walkley and Black, 1934). Estimation of pH value was made in 1 : 2 soil water suspension using Beckman glass electrode pH meter. Collin's calcimeter (Wright, 1934) was employed for the determination of carbonates.

RESULTS AND DISCUSSIONS

TABLE 1

Analysis of different soil groups

Sl. No.			Soil groups Alluvial	Red	Black	Laterite	Hilly and forest
1.	Number of samples analysed		27	11	10	6	6
2.	pH range	0–9″	3·8–8·1	3·7–7·9	6·0–8·3	3·7–8·0	4·6–7·6
		9–18″	3·7–8·2	3·9–8·0	6·4–8·4	3·9–8·1	3·9–7·6
3	Range of $CaCO_3$%	0–9″	Nil–34·048	Nil	Nil–6·175	Nil–1·083	Nil
		9–18″	Nil–33·516	Nil	Nil–6·270	Nil–0·950	Nil
4.	Range of Organic C%	0–9″	0·172–2·200	0·285–4·53	0·355–2·56	0·339–3·18	0·738–2·18
		9–18″	0·101–1·64	0·189–4·12	0·293–1·93	0·279–2·29	0·472–1·71
5.	Range of recovery of C	0–9″	70–92	79–92	78–85	77–89	77–89
		9–18″	68–92	78–94	78–86	77–85	78–86
6.	Mean carbon percentage recovery	0–9″	83·36±5·94	85·32±3·75	82·84±2·34	83·39±4·75	84·34±3·98
		9–18″	80·84±6·19	81·11±3·60	81·02±2·36	81·92±4·23	82·72±2·96
7.	Mean carbon recovery factor	0–9″	1·199	1·172	1·210	1·190	1·180
		9–18″	1·237	1·240	1·230	1·220	1·200

The results show a variation in the recovery of organic carbon as well as in the mean recovery of organic carbon for different soil groups. In the order of high to low values of recovery of organic carbon the soils may be arranged as follows :

(*a*) 0–9″ Red soils>Hilly and Forest soils>Laterite soils>Aluvial soils >and Black soils.

(*b*) 9-18″ Hilly and Forest soils>Laterite soils>Red soils>Black soils>and Alluvial soils.

High mean recovery was obtained in hilly and forest soils and red soils. It appeared that the organic matter present in the former was comparatively much fresh and therefore susceptible to oxidation. The red soils were rich in Fe_2O_3 and Al_2O_3 and so some quantity of iron and alluminium may be present in the divalent forms. It is interesting to record that in black and alluvial soils where the mean recovery of organic carbon has been found as lowest were rich in liming materials also. The present liming material may retard the process of oxidation by protecting the organic matter particles with $CaSO_4$ granules formed in the reaction.

This opinion was expressed by Dewan *et al.* (1959). Gypsum is also known to exert detrimental effect on the process of microbial oxidation of organic matter in the soil (Ames and Richard, 1917). It may be noted that red and hilly and forest soils were conspicuous in the absence of $CaCO_3$.

The percentage mean recovery of organic carbon was less for subsurface soils than surface soils, for all the soil groups investigated (Table 1). This indicated that deep layer organic matter is more humified and resistant to oxidation.

The mean recovery precentage data found for all the soils was in considerable difference from the mean value of 77% reported by Walkley. For computing values for total organic carbon the mean percentage recovery or the recovery factor as reported in Table 1 should be employed with due regard to the soil order spectrum of the Indian soils. However, taking into account large values of standard errors obtained, it is preferable where the conditions may permit to work out this value for the concerned soil region or area and then use it.

SUMMARY AND CONCLUSION

A large veriety of soil samples from surface as well as subsurface, representing major soil types of India were collected. These were analysed for pH, $CaCO_3$ and organic C content by Walkley–Black method and dry combustion method. The mean recovery of organic carbon found by Walkley-Black method, was in considerable difference from the conventional value of 77%. It was found to vary with the soil group and for all the soils analysed, it was less for subsurface than surface samples.

Total estimation of organic carbon should be made on the basis of the obtained factors. Since, these values are accompained with marked variations in standard errors, wherever it is feasible the recovery factor for the stipulated soil region may be found out and applied.

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